Introduction to modelization of thick and heterogeneous plates

Arthur Lebée, Karam Sab

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Laboratoire Navier (UMR CNRS 8205)
Université Paris-Est - École des Ponts ParisTech - IFSTTAR

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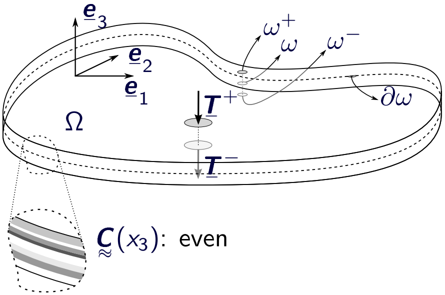
The 3D Problem

$$\begin{cases} \sigma^t_{ij,j} = 0 \quad \text{on } \Omega. \\ \sigma^t_{ij} = C_{ijkl}(x_3)\varepsilon^t_{kl} \quad \text{on } \Omega. \\ \sigma^t_{i3} = \pm T^\pm_i \quad \text{on } \omega^\pm. \\ \varepsilon^t_{ij} = u^t_{(i,j)} \quad \text{on } \Omega. \\ u^t_i = 0 \quad \text{on } \partial\omega\times]-t/2, t/2[\end{cases}$$

- monoclinic $\underset{\approx}{\boldsymbol{C}}$:
  $C_{\alpha\beta\gamma 3} = C_{333\alpha} = 0$
- symmetrically laminated plate
- symmetric transverse load
  $\underline{\boldsymbol{T}}^\pm = \frac{p_3}{2}\underline{\boldsymbol{e}}_3$



$\Rightarrow$ pure bending:

- u_3 and $\sigma_{\alpha 3}$ even / x_3
- u_α, $\sigma_{\alpha\beta}$ and σ_{33} odd / x_3

Building a plate model?

For typical width L let $t \to 0$

- Solve a 2D problem
- "fair" 3D displacement localization
- "fair" 3D stress localization

Some energy principles...

- Statically admissible fields:

$$SA^{3D} = \left\{ \sigma_{ij} \quad / \quad \sigma_{ij,j} = 0 \quad \text{and} \quad \underset{\sim}{\boldsymbol{\sigma}} \left(\pm\tfrac{t}{2}\right) \cdot \pm \underline{\boldsymbol{e}}_3 = \tfrac{P_3}{2} \underline{\boldsymbol{e}}_3 \right\}$$

$$SA^{3D,0} \Leftrightarrow P_3 = 0$$

- Kinematically admissible fields:

$$KA^{3D} = \left\{ u_i \quad / \quad u_i = 0 \text{ on } \partial\omega \times \left]-\tfrac{t}{2}, \tfrac{t}{2}\right[\right\}$$

$$KA^{3D,0} = KA^{3D}$$

- Orthogonality between $KA^{3D,0}$ and $SA^{3D,0}$:

$$\forall \underline{\boldsymbol{u}} \in KA^{3D,0}, \underset{\sim}{\boldsymbol{\sigma}} \in SA^{3D,0}, \quad \int_\Omega \underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{u}}) d\Omega = 0$$

Some energy principles...

- Potential energy:

$$\underline{\boldsymbol{u}}^{3D} = \underset{\underline{\boldsymbol{u}} \in KA^{3D}}{\operatorname{argmin}} \left\{ W^{3D}\left(\underset{\sim}{\varepsilon}(\underline{\boldsymbol{u}})\right) - \int_\omega P_3 \frac{u_3^+ + u_3^-}{2} d\omega \right\}$$

where $W^{3D} = \frac{1}{2}\int_\Omega \underset{\sim}{\varepsilon}(\underline{\boldsymbol{u}}) : \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}(\underline{\boldsymbol{u}}) d\Omega$

- Complementary energy:

$$\underset{\sim}{\boldsymbol{\sigma}}^{3D} = \underset{\underset{\sim}{\boldsymbol{\sigma}} \in SA^{3D}}{\operatorname{argmin}} \left\{ W^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}\right) \right\} \quad \text{where} \quad W^{*3D} = \frac{1}{2}\int_\Omega \underset{\sim}{\boldsymbol{\sigma}} : \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\boldsymbol{\sigma}} d\Omega$$

$\underset{\approx}{\boldsymbol{S}} = \underset{\approx}{\boldsymbol{C}}^{-1}$

The 2-energy principle

$$\forall \underline{\hat{\boldsymbol{u}}} \in KA^{3D}, \quad \forall \underset{\sim}{\hat{\boldsymbol{\sigma}}} \in SA^{3D}:$$

$$W^{3D}\left(\underset{\sim}{\varepsilon}\left(\underline{\hat{\boldsymbol{u}}}\right) - \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\hat{\boldsymbol{\sigma}}}\right) = W^{*3D}\left(\underset{\sim}{\hat{\boldsymbol{\sigma}}} - \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}\left(\underline{\hat{\boldsymbol{u}}}\right)\right)$$

$$= W^{*3D}\left(\underset{\sim}{\hat{\boldsymbol{\sigma}}} - \underset{\sim}{\boldsymbol{\sigma}}^{3D}\right) + W^{3D}\left(\underset{\sim}{\varepsilon}\left(\underline{\hat{\boldsymbol{u}}} - \underline{\boldsymbol{u}}^{3D}\right)\right)$$

$\Rightarrow W^{*3D}\left(\underset{\sim}{\hat{\boldsymbol{\sigma}}} - \underset{\approx}{\boldsymbol{C}} : \underset{\sim}{\varepsilon}\left(\underline{\hat{\boldsymbol{u}}}\right)\right)$ provides an error estimate in terms of constitutive equation.

Prager and Synge (1947); Morgenstern (1959); Braess et al. (2010)

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Natural scaling of the stress

$$SA^{3D}\begin{cases}\sigma_{\alpha\beta,\beta}+\sigma_{\alpha 3,3}=0\\ \sigma_{\alpha 3,\alpha}+\sigma_{33,3}=0\\ \sigma_{33}(\pm t/2)=\pm P_3/2\\ \sigma_{\alpha 3}(\pm t/2)=0\end{cases}\quad\Rightarrow\quad\begin{cases}\sigma_{\alpha 3}=-\displaystyle\int_{-t/2}^{x_3}\sigma_{\alpha\beta,\beta}dz\\ \sigma_{33}=-\displaystyle\int_{t/2}^{x_3}\sigma_{\alpha 3,\alpha}dz-P_3/2\end{cases}$$

$$\sigma_{\alpha\beta}\sim t^0\quad\Rightarrow\quad\sigma_{\alpha 3}\sim t^1,\quad\sigma_{33}\sim t^2\quad\text{and}\quad P_3\sim t^2$$

From 3D equilibrium to 2D

Plate generalized stresses:

$$\begin{cases} M_{\alpha\beta}(x_1, x_2) = \langle x_3 \sigma_{\alpha\beta} \rangle & \sim t^2 \\ Q_\alpha (x_1, x_2) = \langle \sigma_{\alpha 3} \rangle & \sim t^2 \end{cases} \qquad \langle \bullet \rangle = \int_{-\frac{t}{2}}^{\frac{t}{2}} \bullet \, dx_3$$

2D equilibrium equations:

$$\begin{cases} \langle \sigma_{\alpha 3,\alpha} + \sigma_{33,3} \rangle = 0 \\ \langle x_3 (\sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3}) \rangle = 0 \end{cases} \Rightarrow \begin{cases} Q_{\alpha,\alpha} + P_3 = 0 \\ M_{\alpha\beta,\beta} - Q_\alpha = 0 \end{cases}$$

Boussinesq (1871); Mindlin (1951)...

"Kirchhoff's assumption"

At leading order in t:

$$\begin{cases} \varepsilon_{\alpha\beta} = x_3 K_{\alpha\beta} \quad \text{where} \quad K_{\alpha\beta} = -U_{3,\alpha\beta} \\ \sigma_{i3} \simeq 0 + O(t^1) \quad \text{plane stress} \end{cases}$$

$\varepsilon_{33} \neq 0$!!

In-plane stress:

$$\sigma_{\alpha\beta} = C^{\sigma}_{\alpha\beta\gamma\delta}\varepsilon_{\delta\gamma} = x_3 C^{\sigma}_{\alpha\beta\gamma\delta} K_{\delta\gamma}$$

where $C^{\sigma}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - \frac{C_{\alpha\beta 33} C_{33\gamma\delta}}{C_{3333}}$:

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{pmatrix}$$

The Kirchhoff-Love plate problem

Bending constitutive equation:

$$M_{\alpha\beta} = \langle x_3 \sigma_{\alpha\beta} \rangle = \left\langle x_3^2 C^{\sigma}_{\alpha\beta\gamma\delta} \right\rangle K_{\delta\gamma} = -D_{\alpha\beta\gamma\delta} U_{3,\delta\gamma}$$

$$\underset{\approx}{\boldsymbol{D}} = \frac{t^3}{12} \underset{\approx}{\boldsymbol{C}}^{\sigma}$$

Statically admissible fields:

$$SA^{KL} : \{ M_{\alpha\beta} / M_{\alpha\beta,\alpha\beta} + P_3 = 0 \}$$

Kinematically compatible fields:

$$KA^{KL} : \left\{ U_3 / U_3 = 0 \quad \text{and} \quad U_{3,n} = 0 \text{ on } \partial\omega, \underline{\boldsymbol{n}} \text{ outer normal to } \omega \right\}$$

always "hard support": $U_3 = 0$ on $\partial\omega \Rightarrow U_{3,t} = 0$ on $\partial\omega$

Building SA^{3D} fields

$$
\begin{aligned}
\hat{\sigma}_{\alpha\beta} &= x_3 C^{\sigma}_{\alpha\beta\gamma\delta} d_{\delta\gamma\epsilon\zeta} M_{\zeta\epsilon} && = \frac{12x_3}{t^3} M_{\alpha\beta} && \sim t^0 \\
\hat{\sigma}_{\alpha 3} &= -\int_{-t/2}^{x_3} \sigma_{\alpha\beta,\beta} dz && = \frac{3}{2t}\left(1 - \frac{4x_3^2}{t^2}\right) M_{\alpha\beta,\beta} && \sim t^1 \\
\hat{\sigma}_{33} &= -\int_{-t/2}^{x_3} \sigma_{\alpha 3,\alpha} dz - P_3/2 && = \frac{3x_3}{2t}\left(1 - \frac{4x_3^2}{3t^2}\right) P_3 && \sim t^2
\end{aligned}
$$

$$
\Leftrightarrow \quad \hat{\underset{\sim}{\boldsymbol{\sigma}}} = \underset{\approx}{\mathbf{s}}^M(x_3) : \underset{\sim}{\boldsymbol{M}} + \underset{\approx}{\mathbf{s}}^Q(x_3) \cdot \left(\underset{\sim}{\boldsymbol{M}} \cdot \underline{\boldsymbol{\nabla}}\right) + \underset{\sim}{\mathbf{s}}^{P_3}(x_3) P_3
$$

$$
\hat{\underset{\sim}{\boldsymbol{\sigma}}} \in SA^{3D}
$$

Building KA^{3D} fields

Strains

$$\hat{\underset{\sim}{\varepsilon}} = \begin{cases} \hat{\varepsilon}_{\alpha\beta} = x_3 d_{\alpha\beta\gamma\delta} M_{\gamma\delta} & = S_{\alpha\beta\gamma\delta}\hat{\sigma}_{\delta\gamma} + \cancel{S_{\alpha\beta 33}\hat{\sigma}_{33}} \\ \hat{\varepsilon}_{\alpha 3} = \dfrac{3}{4Gt}\left(1 - \dfrac{4x_3^2}{t^2}\right) M_{\alpha\beta,\beta} & = 2S_{\alpha 3\beta 3}\hat{\sigma}_{\beta 3} \\ \hat{\varepsilon}_{33} = -\dfrac{12\nu x_3}{Et^3} M_{\alpha\alpha} & = S_{33\alpha\beta}\hat{\sigma}_{\alpha\beta} + \cancel{S_{3333}\hat{\sigma}_{33}} \end{cases}$$

Building KA^{3D} fields

Integration

$$\hat{u}_3 = \int^{x_3} \hat{\varepsilon}_{33}(z)dz + U_3 = \underbrace{\mathbb{u}^M_{3\alpha\beta}(x_3)M_{\beta\alpha}}_{\sim t^1} + \underbrace{U_3}_{\sim t^{-1}}$$

where: $\mathbb{u}^M_{3\alpha\beta}(x_3) = \dfrac{\nu}{2Et}\left(\dfrac{12x_3^2}{t^2} - 1\right)\delta_{\alpha\beta}$ and $\left\langle \mathbb{u}^M_{3\alpha\beta}(x_3) \right\rangle = 0$

Building KA^{3D} fields

Integration

$$\hat{u}_3 = \int^{x_3} \hat{\varepsilon}_{33}(z)dz + U_3 = \underbrace{\mathbb{u}^{M}_{3\alpha\beta}(x_3)M_{\beta\alpha}}_{\sim t^1} + \underbrace{U_3}_{\sim t^{-1}}$$

$$\hat{u}_\alpha = \int^{x_3} 2\hat{\varepsilon}_{\alpha 3}(z) - \hat{u}_{3,\alpha}dz = \underbrace{\mathbb{u}^{M\otimes\nabla}_{\alpha\beta\gamma\delta}(x_3)M_{\delta\gamma,\beta}}_{\sim t^2} - \underbrace{x_3 U_{3,\alpha}}_{\sim t^0}$$

where:

$$\mathbb{u}^{M\otimes\nabla}_{\alpha\beta\gamma\delta}(x_3)M_{\delta\gamma\beta} = \frac{x_3}{2Et}\left(6(1+\nu)\left(1-\frac{4x_3^2}{3t^2}\right)M_{\alpha\beta,\beta} + \nu\left(1-\frac{4x_3^2}{t^2}\right)M_{\beta\beta,\alpha}\right)$$

and $\left\langle \mathbb{u}^{M\otimes\nabla}_{\alpha\beta\gamma\delta}(x_3)\right\rangle = 0$

Building KA^{3D} fields

Integration

$$\hat{u}_3 = \int^{x_3} \hat{\varepsilon}_{33}(z)dz + U_3 = \underbrace{\mathbb{u}^{M}_{3\alpha\beta}(x_3)M_{\beta\alpha}}_{\sim t^1} + \underbrace{U_3}_{\sim t^{-1}}$$

$$\hat{u}_\alpha = \int^{x_3} 2\hat{\varepsilon}_{\alpha 3}(z) - \hat{u}_{3,\alpha}dz = \underbrace{\mathbb{u}^{M\otimes\nabla}_{\alpha\beta\gamma\delta}(x_3)M_{\delta\gamma,\beta}}_{\sim t^2} - \underbrace{x_3 U_{3,\alpha}}_{\sim t^0}$$

$$\Rightarrow \quad \hat{\underline{\boldsymbol{u}}} = U_3 \underline{\boldsymbol{e}}_3 - x_3 U_{3,\alpha} \underline{\boldsymbol{e}}_\alpha + \mathbb{u}^{M}_{3\alpha\beta}(x_3)M_{\beta\alpha}\underline{\boldsymbol{e}}_3 + \mathbb{u}^{M\otimes\nabla}_{\alpha\beta\gamma\delta}(x_3)M_{\delta\gamma,\beta}\underline{\boldsymbol{e}}_\alpha$$

with:

$$\hat{\underset{\approx}{\varepsilon}} = \underset{\approx}{\varepsilon}\left(\hat{\underline{\boldsymbol{u}}}\right)$$

Application of the Two-Energy principle

Consider U_3^{KL} and $\underset{\sim}{\boldsymbol{M}}^{KL}$ the solution of the Kirchhoff-Love plate problem and define:

$$\underset{\sim}{\boldsymbol{\sigma}}^{KL} = \underset{\approx}{\mathbb{s}}^{M}(x_3) : \underset{\sim}{\boldsymbol{M}}^{KL} + \underset{\approx}{\mathbb{s}}^{Q}(x_3) \cdot \left(\underset{\sim}{\boldsymbol{M}}^{KL} \cdot \underline{\boldsymbol{\nabla}} \right) + \underset{\approx}{\mathbb{s}}^{P_3}(x_3) P_3$$

$$\underline{\boldsymbol{u}}^{KL} = U_3^{KL} \underline{\boldsymbol{e}}_3 - x_3 U_{3,\alpha}^{KL} \underline{\boldsymbol{e}}_\alpha + \mathbb{u}_{3\alpha\beta}^{M}(x_3) M_{\beta\alpha}^{KL} \underline{\boldsymbol{e}}_3 + \mathbb{u}_{\alpha\beta\gamma\delta}^{M\otimes\nabla}(x_3) M_{\delta\gamma,\beta}^{KL} \underline{\boldsymbol{e}}_\alpha$$

We have:

$$\underset{\sim}{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{u}}^{KL}) - \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\boldsymbol{\sigma}}^{KL} = \mathbb{s}_{33}^{P_3} P_3 \begin{pmatrix} S_{\alpha\beta 33} & & 0 \\ & & 0 \\ 0 & 0 & S_{3333} \end{pmatrix} \sim t^2$$

Application of the Two-Energy principle

$$\underset{\sim}{\varepsilon}(\underline{\boldsymbol{u}}^{KL}) - \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\boldsymbol{\sigma}}^{KL} \sim t^2 \quad \Rightarrow \quad W^{3D}\left(\underset{\sim}{\varepsilon}(\underline{\boldsymbol{u}}^{KL}) - \underset{\approx}{\boldsymbol{S}} : \underset{\sim}{\boldsymbol{\sigma}}^{KL}\right) \sim t^5$$

Would lead to a relative error in t^2...

$\underset{\sim}{\boldsymbol{\sigma}}^{KL} \in SA^{3D}$ **but** $\underline{\boldsymbol{u}}^{KL} \notin KA^{3D}$

At best: relative error in $t^{1/2}$...

Braess et al. (2010) among many others!

Reissner's original plate model (1945)

min of W^{*3D}

Let us consider:

$$\hat{\underset{\sim}{\boldsymbol{\sigma}}} = \underset{\approx}{\mathbf{s}}^{M}(x_3) : \underset{\sim}{\boldsymbol{M}} + \underset{\approx}{\mathbf{s}}^{Q}(x_3) \cdot \underline{\boldsymbol{Q}} + \underset{\sim}{\mathbf{s}}^{P_3}(x_3) P_3$$

with

$$SA^{RM} = \left\{ (\underset{\sim}{\boldsymbol{M}}, \underline{\boldsymbol{Q}}) \;/\; Q_{\alpha,\alpha} + P_3 = 0 \quad \text{and} \quad M_{\alpha\beta,\beta} - Q_\alpha = 0 \quad \text{on} \quad \omega \right\}$$

$$W^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}^{3D}\right) \leq \min_{(\underset{\sim}{\boldsymbol{M}}, \underline{\boldsymbol{Q}}) \in SA^{RM}} W^{*3D}\left(\hat{\underset{\sim}{\boldsymbol{\sigma}}}\right) \leq W^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}^{KL}\right)$$

is a better approximation of $W^{*3D}\left(\underset{\sim}{\boldsymbol{\sigma}}^{3D}\right)$

Reissner's original plate model (1945)

Dualization

$$\left\{\begin{array}{ll} Q_{\alpha,\alpha} + P_3 = 0 & \times U_3 \\ M_{\alpha\beta,\beta} - Q_\alpha = 0 & \times \varphi_\alpha \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{llll} Q_\alpha & \leftrightarrow & \gamma_\alpha & = \varphi_\alpha + U_{3,\alpha} \\ M_{\alpha\beta} & \leftrightarrow & \chi_{\alpha\beta} & = \varphi_{(\alpha,\beta)} \end{array}\right.$$

$$KA^{RM} = \{(U_3, \varphi_\alpha) \;/\; U_3 = 0 \quad \text{and} \quad \varphi_\alpha = 0 \text{ on } \partial\omega\}$$

Reissner's original plate model (1945)

Constitutive equation

$$W^{*RM}\left(\underset{\sim}{\boldsymbol{M}}, \underline{\boldsymbol{Q}}\right) = W^{*3D}\left(\hat{\underset{\sim}{\boldsymbol{\sigma}}}\right)$$

$$= \frac{1}{2}\int_\omega \begin{pmatrix} \underset{\sim}{\boldsymbol{M}} \\ \underline{\boldsymbol{Q}} \\ P_3 \end{pmatrix}^T \begin{pmatrix} \underset{\approx}{\boldsymbol{d}} & 0 & \frac{6\nu}{5Et}\underset{\sim}{\boldsymbol{\delta}} \\ 0 & \frac{6}{5Gt}\underset{\sim}{\boldsymbol{\delta}} & 0 \\ \frac{6\nu}{5Et}\underset{\sim}{\boldsymbol{\delta}} & 0 & \frac{17t}{140E} \end{pmatrix} \begin{pmatrix} \underset{\sim}{\boldsymbol{M}} \\ \underline{\boldsymbol{Q}} \\ P_3 \end{pmatrix} d\omega$$

$$\Rightarrow \quad \begin{cases} \underbrace{\chi_{\alpha\beta}}_{\sim t^{-1}} = \underbrace{d_{\alpha\beta\gamma\delta}M_{\delta\gamma}}_{\sim t^{-1}} + \underbrace{\frac{6\nu}{5Et}\delta_{\alpha\beta}P_3}_{\sim t^{1}} \\ \gamma_\alpha = \dfrac{6}{5Gt}Q_\alpha \sim t^1 \end{cases}$$

The contribution of P_3 is almost always dropped in the literature

Contents

Field Localization

Following the same procedure leads to:

$$\hat{\underset{\sim}{\sigma}} = \underset{\approx}{\mathbb{s}}^M(x_3) : \underset{\sim}{\boldsymbol{M}} + \underbrace{\underset{\approx}{\mathbb{s}}^R(x_3) \vdots \underset{\sim}{\boldsymbol{M}} \otimes \underline{\boldsymbol{\nabla}}}_{\sim t^1} + \underbrace{\underset{\approx}{\mathbb{s}}^T(x_3) \vdots \underset{\sim}{\boldsymbol{M}} \otimes \underline{\boldsymbol{\nabla}}^2 + \underset{\approx}{\mathbb{s}}^{P_3}(x_3) P_3}_{\sim t^2}$$

$$\hat{\underline{\boldsymbol{u}}} = U_3 \underline{\boldsymbol{e}}_3 - x_3 U_{3,\alpha} \underline{\boldsymbol{e}}_\alpha + \mathbb{u}^M_{3\alpha\beta}(x_3) M_{\beta\alpha} \underline{\boldsymbol{e}}_3 + \mathbb{u}^R_{\alpha\beta\gamma\delta}(x_3) M_{\delta\gamma,\beta} \underline{\boldsymbol{e}}_\alpha$$

where $\boldsymbol{R} = \underset{\sim}{\boldsymbol{M}} \otimes \underline{\boldsymbol{\nabla}}$ and $\underset{\approx}{\boldsymbol{T}} = \boldsymbol{R} \otimes \underline{\boldsymbol{\nabla}}$

... Kirchhoff-Love error estimates still hold.

Building SA^{3D} fields

$$
\begin{aligned}
\hat{\sigma}_{\alpha\beta} &= x_3 C^{\sigma}_{\alpha\beta\gamma\delta} d_{\delta\gamma\epsilon\zeta} M_{\zeta\epsilon} && = \mathbb{s}^{M}_{\alpha\beta\gamma\delta}(x_3) M_{\delta\gamma} \\
\hat{\sigma}_{\alpha 3} &= -\int_{-t/2}^{x_3} \sigma_{\alpha\beta,\beta} dz && = \mathbb{s}^{R}_{\alpha 3\beta\gamma\delta}(x_3) M_{\delta\gamma,\beta} \\
\hat{\sigma}_{33} &= -\int_{-t/2}^{x_3} \sigma_{\alpha 3,\alpha} dz - P_3/2 && = \mathbb{s}^{T}_{33\alpha\beta\gamma\delta}(x_3) M_{\delta\gamma,\beta\alpha} + \mathbb{s}^{P_3}_{33}(x_3) P_3
\end{aligned}
$$

The Bending-Gradient constitutive equation

Extending Reissner's approach?:

$$\underset{\sim}{\hat{\boldsymbol{\sigma}}} = \underset{\approx}{\mathbf{s}}^{M}(x_3) : \underset{\sim}{\boldsymbol{M}} + \underset{\frown}{\underset{\sim}{\mathbf{s}}}^{R}(x_3) \vdots \underset{\frown}{\boldsymbol{R}} + \cancel{\underset{\frown}{\underset{\sim}{\mathbf{s}}}^{T}(x_3) \vdots \underset{\approx}{\boldsymbol{T}}} + \cancel{\underset{\sim}{\mathbf{s}}^{P_3}(x_3)} P_3$$

Let us define:

$$\underset{\sim}{\hat{\boldsymbol{\sigma}}}^{*} = \underset{\approx}{\mathbf{s}}^{M}(x_3) : \underset{\sim}{\boldsymbol{M}} + \underset{\frown}{\underset{\sim}{\mathbf{s}}}^{R}(x_3) \vdots \underset{\frown}{\boldsymbol{R}}$$

with: $\underset{\sim}{\hat{\boldsymbol{\sigma}}}^{*} \cdot \underline{\boldsymbol{\nabla}} = 0 + O(t^1)$ only.

$$W^{*BG}\left(\underset{\sim}{\boldsymbol{M}}, \underset{\frown}{\boldsymbol{R}}\right) = W^{*3D}\left(\underset{\sim}{\hat{\boldsymbol{\sigma}}}^{*}\right) = \frac{1}{2}\int_{\omega} \left(\underset{\sim}{\boldsymbol{M}} : \underset{\approx}{\boldsymbol{d}} : \underset{\sim}{\boldsymbol{M}} + \underset{\frown}{\boldsymbol{R}} \vdots \underset{\frown}{\underset{\frown}{\boldsymbol{f}}} \vdots \underset{\frown}{\boldsymbol{R}}\right) d\omega$$

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG: } \begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases} \qquad \text{RM: } \begin{cases} Q_{\alpha} = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases}$$

- Mechanical meaning of $\underset{\frown}{\boldsymbol{R}}$

$$Q_\alpha = R_{\alpha\beta\beta} \Leftrightarrow \begin{cases} Q_1 = R_{111} + R_{122} = M_{11,1} + M_{12,2} \\ Q_2 = R_{121} + R_{222} = M_{21,1} + M_{22,2} \end{cases}$$

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG:} \left\{ \begin{array}{ll} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} & \times \Phi_{\alpha\beta\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 & \times U_3 \end{array} \right. \qquad \text{RM:} \left\{ \begin{array}{ll} Q_\alpha = M_{\alpha\beta,\beta} & \times \varphi_\alpha \\ Q_{\alpha,\alpha} + P_3 = 0 & \times U_3 \end{array} \right.$$

The Bending-Gradient theory for thick plates

- Equilibrium equations:

BG: $\begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} & \times\ \Phi_{\alpha\beta\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 & \times\ U_3 \end{cases}$ RM: $\begin{cases} Q_\alpha = M_{\alpha\beta,\beta} & \times\ \varphi_\alpha \\ Q_{\alpha,\alpha} + P_3 = 0 & \times\ U_3 \end{cases}$

	Equilibrium	Work of internal forces	Work on Boundary
BG:	$\begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 \end{cases}$	$M_{\alpha\beta}\Phi_{\alpha\beta\epsilon,\epsilon} +$ $R_{\alpha\beta\gamma}\left(\Phi_{\alpha\beta\gamma} + I_{\alpha\beta\gamma\epsilon}U_{3,\epsilon}\right)$	$M_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_\gamma +$ $R_{\alpha\beta\beta}n_\alpha U_3$
RM:	$\begin{cases} Q_\alpha = M_{\alpha\beta,\beta} \\ Q_{\alpha,\alpha} + P_3 = 0 \end{cases}$	$M_{\alpha\beta}\,\varphi_{(\alpha,\beta)} +$ $Q_\alpha\left(\varphi_\alpha + U_{3,\alpha}\right)$	$M_{\alpha\beta}n_\beta\varphi_\alpha +$ $Q_\alpha n_\alpha U_3$

$I_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right)$ identity for in-plane elasticity

The Bending-Gradient theory for thick plates

- Equilibrium equations:

$$\text{BG:} \begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} & \times \Phi_{\alpha\beta\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 & \times U_3 \end{cases} \qquad \text{RM:} \begin{cases} Q_{\alpha} = M_{\alpha\beta,\beta} & \times \varphi_{\alpha} \\ Q_{\alpha,\alpha} + P_3 = 0 & \times U_3 \end{cases}$$

	Equilibrium	Work of internal forces	Work on Boundary
BG:	$R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$	$M_{\alpha\beta}\Phi_{\alpha\beta\epsilon,\epsilon} +$	$M_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_{\gamma} +$
	$R_{\alpha\beta\beta,\alpha} + P_3 = 0$	$R_{\alpha\beta\gamma}\left(\Phi_{\alpha\beta\gamma} + I_{\alpha\beta\gamma\epsilon}U_{3,\epsilon}\right)$	$R_{\alpha\beta\beta}n_{\alpha}U_3$
RM:	$Q_{\alpha} = M_{\alpha\beta,\beta}$	$M_{\alpha\beta}\,\varphi_{(\alpha,\beta)} +$	$M_{\alpha\beta}n_{\beta}\varphi_{\alpha} +$
	$Q_{\alpha,\alpha} + P_3 = 0$	$Q_{\alpha}\left(\varphi_{\alpha} + U_{3,\alpha}\right)$	$Q_{\alpha}n_{\alpha}U_3$

When the plate is homogeneous, BG is exactly turned into RM
(loss of invertibility of $\underset{\approx}{f}$)

The Bending-Gradient theory for thick plates

- Equilibrium equations:

BG: $\begin{cases} R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma} & \times\ \Phi_{\alpha\beta\gamma} \\ R_{\alpha\beta\beta,\alpha} + P_3 = 0 & \times\ U_3 \end{cases}$ RM: $\begin{cases} Q_{\alpha} = M_{\alpha\beta,\beta} & \times\ \varphi_{\alpha} \\ Q_{\alpha,\alpha} + P_3 = 0 & \times\ U_3 \end{cases}$

	Equilibrium	Work of internal forces	Work on Boundary
BG:	$R_{\alpha\beta\gamma} = M_{\alpha\beta,\gamma}$	$M_{\alpha\beta}\Phi_{\alpha\beta\epsilon,\epsilon} +$	$M_{\alpha\beta}\Phi_{\alpha\beta\gamma}n_{\gamma} +$
	$R_{\alpha\beta\beta,\alpha} + P_3 = 0$	$R_{\alpha\beta\gamma}\left(\Phi_{\alpha\beta\gamma} + I_{\alpha\beta\gamma\epsilon}U_{3,\epsilon}\right)$	$R_{\alpha\beta\beta}n_{\alpha}U_3$
RM:	$Q_{\alpha} = M_{\alpha\beta,\beta}$	$M_{\alpha\beta}\,\varphi_{(\alpha,\beta)} +$	$M_{\alpha\beta}n_{\beta}\varphi_{\alpha} +$
	$Q_{\alpha,\alpha} + P_3 = 0$	$Q_{\alpha}\left(\varphi_{\alpha} + U_{3,\alpha}\right)$	$Q_{\alpha}n_{\alpha}U_3$

Low order interpolation for U_3 and $\underset{\frown}{\mathbf{\Phi}}$!

Local Fields reconstruction

Once the plate problem is solved (U_3^{BG}, $\underset{\frown}{\boldsymbol{\Phi}}^{BG}$, $\underset{\sim}{\boldsymbol{M}}^{BG}$, $\underset{\frown}{\boldsymbol{R}}^{BG}$ known), we suggest the following field reconstruction:

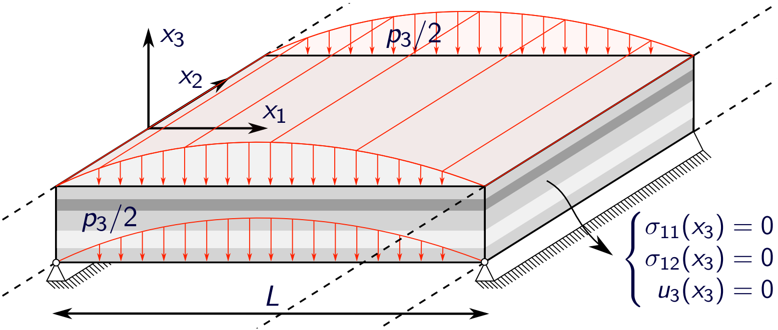
- $\underset{\sim}{\boldsymbol{\sigma}}^{BG} = \underset{\approx}{\mathbb{s}}^{M}(x_3) : \underset{\sim}{\boldsymbol{M}} + \underset{\approx}{\mathbb{s}}^{R}(x_3) \vdots \underset{\frown}{\boldsymbol{R}} + \underset{\approx}{\mathbb{s}}^{T}(x_3) \vdots \underset{\frown}{\boldsymbol{R}} \otimes \underline{\boldsymbol{\nabla}} + \underset{\sim}{\mathbb{s}}^{P_3}(x_3) P_3$
- $\underline{\boldsymbol{u}}^{BG} = U_3 \underline{\boldsymbol{e}}_3 - x_3 U_{3,\alpha} \underline{\boldsymbol{e}}_\alpha + \mathbb{u}^{M}_{3\alpha\beta}(x_3) M_{\beta\alpha} \underline{\boldsymbol{e}}_3 + \mathbb{u}^{R}_{\alpha\beta\gamma\delta}(x_3) R_{\delta\gamma\beta} \underline{\boldsymbol{e}}_\alpha$

Contents

Contents

Pagano's boundary value problem (Pagano, 1969)

CFRP layers with different orientations:



Invariant in x_2-Direction, "periodic" in x_1-Direction

$\Rightarrow$ No boundary layer!

Practical Localization...

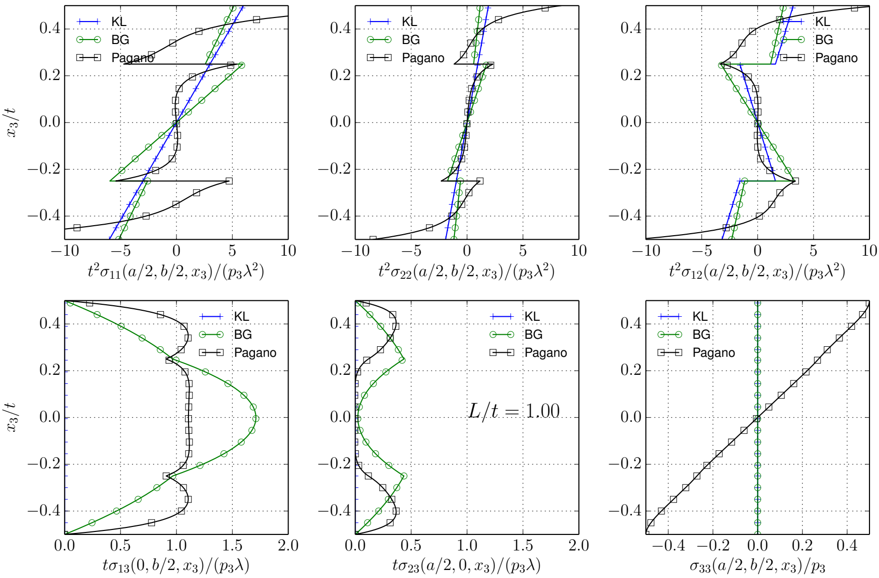
Kirchhoff-Love

- $\underset{\sim}{\boldsymbol{\sigma}}^{KL} = \underset{\approx}{\mathbb{s}}^{M}(x_3) : \underset{\sim}{\boldsymbol{M}}^{KL} + \cancel{\underset{\cong}{\mathbb{s}}^{Q}(x_3) \cdot \left(\underset{\sim}{\boldsymbol{M}}^{KL} \cdot \underline{\boldsymbol{\nabla}}\right)} + \cancel{\underset{\sim}{\mathbb{s}}^{P_3}(x_3) P_3}$
- $\underline{\boldsymbol{u}}^{KL} = U_3^{KL} \underline{\boldsymbol{e}}_3 - x_3 U_{3,\alpha}^{KL} \underline{\boldsymbol{e}}_\alpha + \mathbb{u}_{3\alpha\beta}^{M}(x_3) M_{\beta\alpha}^{KL} \underline{\boldsymbol{e}}_3 + \cancel{\mathbb{u}_{\alpha\beta\gamma\delta}^{M\otimes\nabla}(x_3) M_{\delta\gamma,\beta}^{KL}} \underline{\boldsymbol{e}}_\alpha$

Bending-Gradient

- $\underset{\sim}{\boldsymbol{\sigma}}^{BG} = \underset{\approx}{\mathbb{s}}^{M}(x_3) : \underset{\sim}{\boldsymbol{M}} + \mathbb{s}^{R}(x_3) \vdots \boldsymbol{R} + \cancel{\mathbb{s}^{T}(x_3) \vdots \boldsymbol{R} \otimes \underline{\boldsymbol{\nabla}}} + \cancel{\underset{\sim}{\mathbb{s}}^{P_3}(x_3) P_3}$
- $\underline{\boldsymbol{u}}^{BG} = U_3 \underline{\boldsymbol{e}}_3 - x_3 U_{3,\alpha} \underline{\boldsymbol{e}}_\alpha + \mathbb{u}_{3\alpha\beta}^{M}(x_3) M_{\beta\alpha} \underline{\boldsymbol{e}}_3 + \mathbb{u}_{\alpha\beta\gamma\delta}^{R}(x_3) R_{\delta\gamma\beta} \underline{\boldsymbol{e}}_\alpha$

Stress distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



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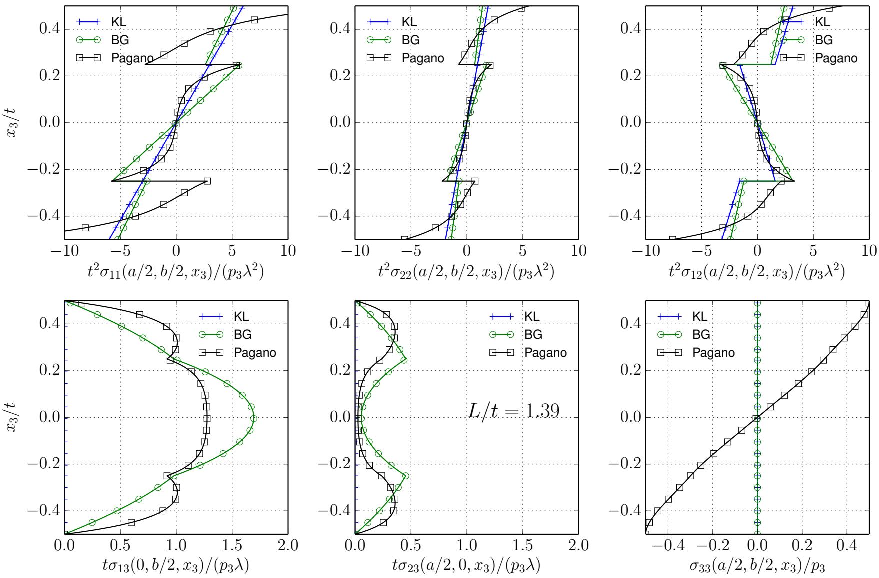
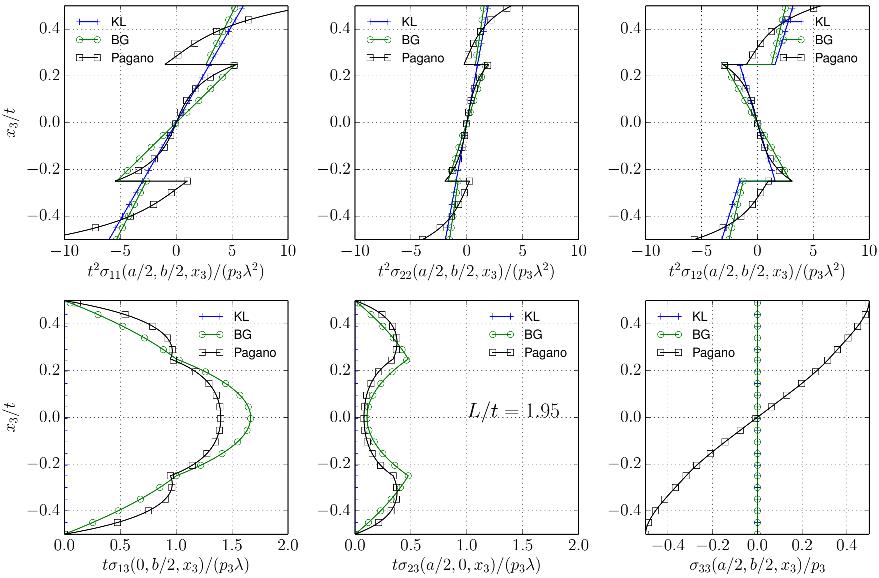


Stress distributions for a $[30°, -30°, 30°]$ stack



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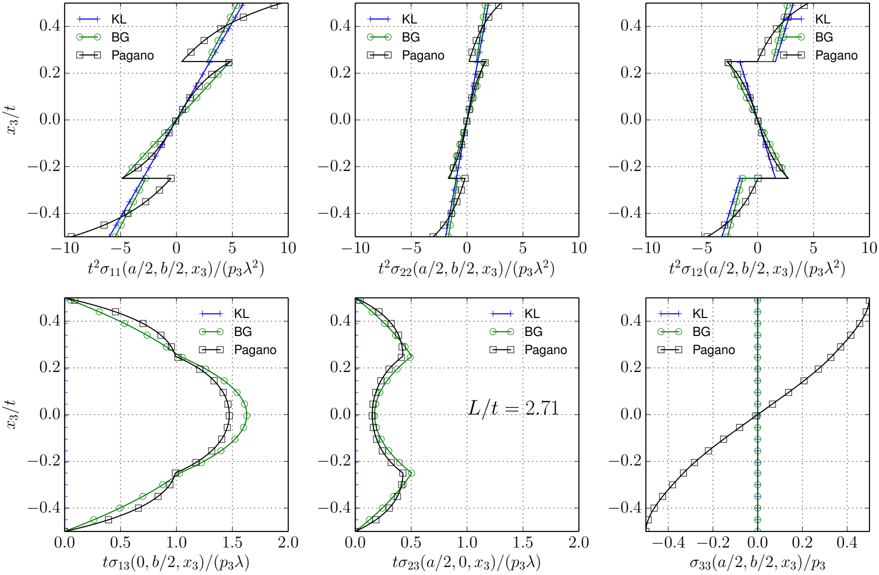

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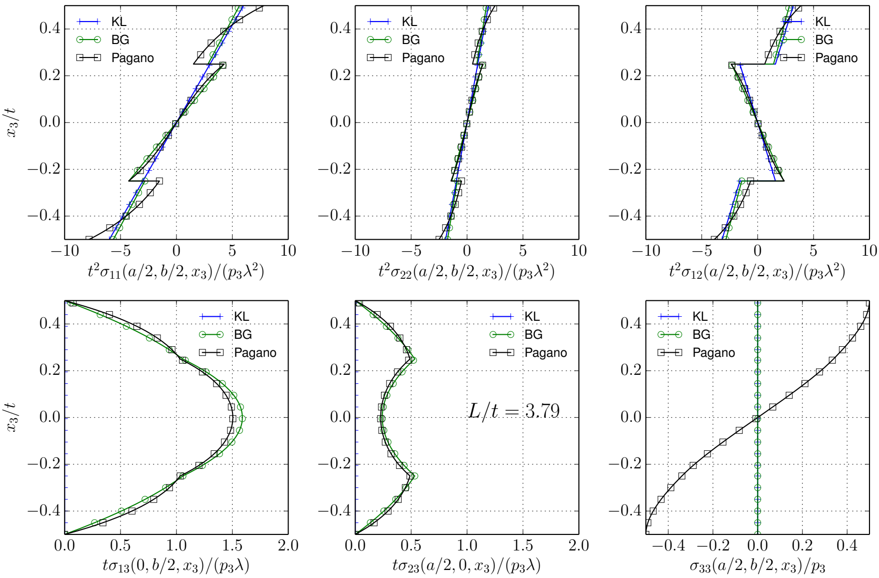

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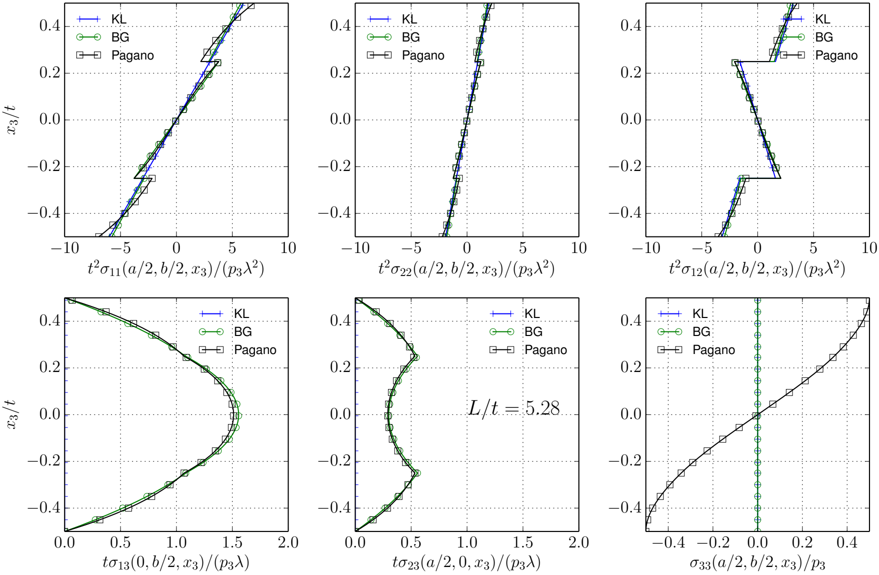

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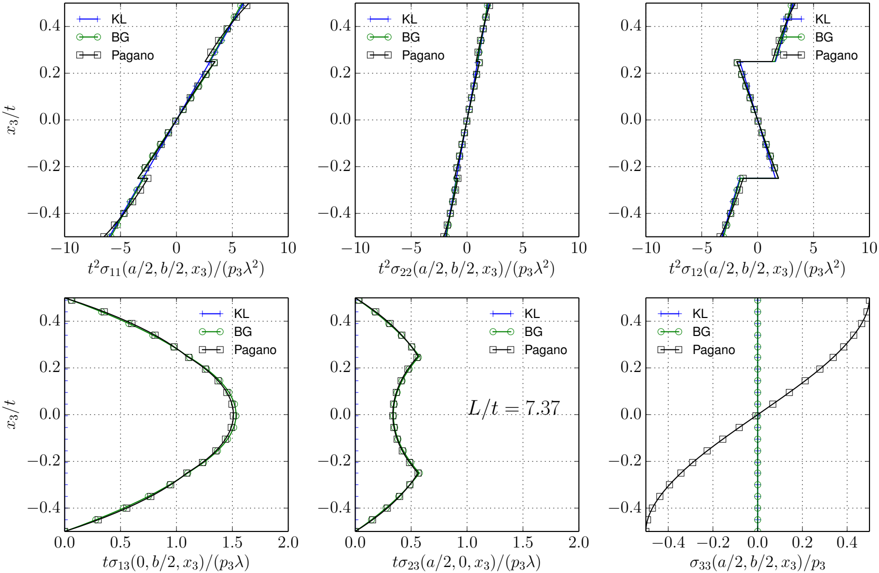

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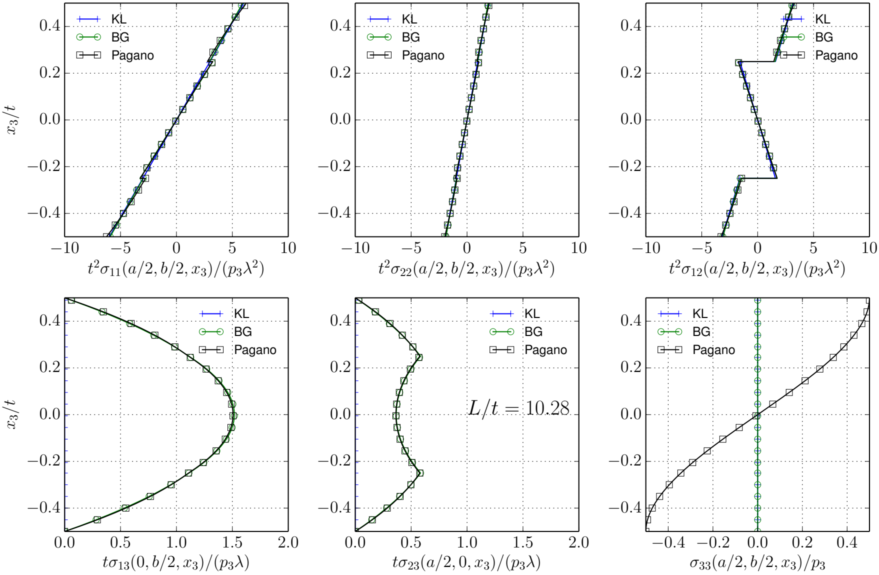

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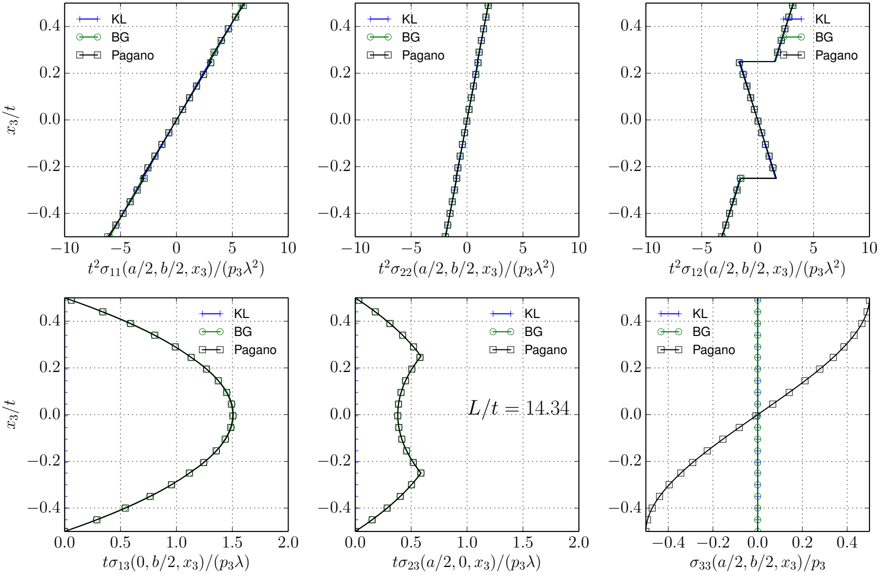

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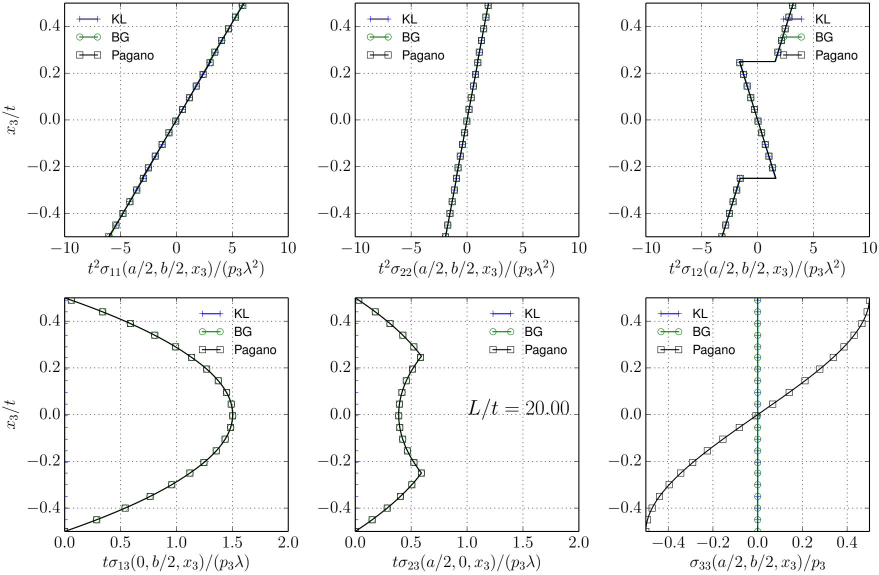
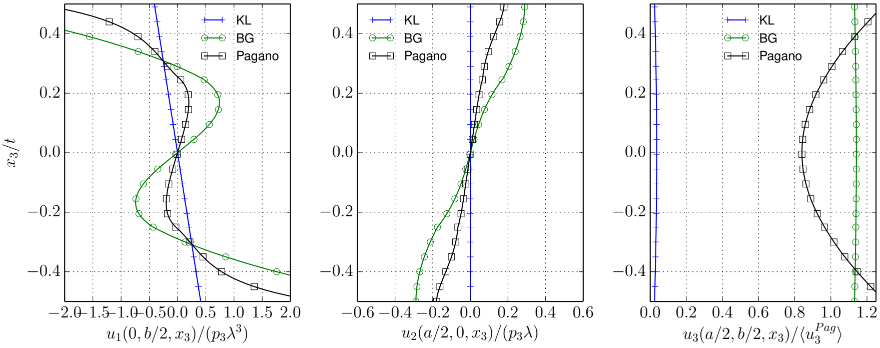
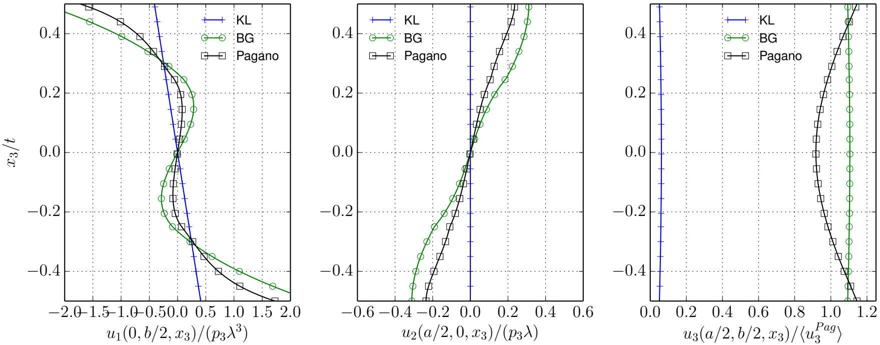


Displacement distributions for a $[30°, -30°, 30°]$ stack



Displacement distributions for a $[30^\circ, -30^\circ, 30^\circ]$ stack



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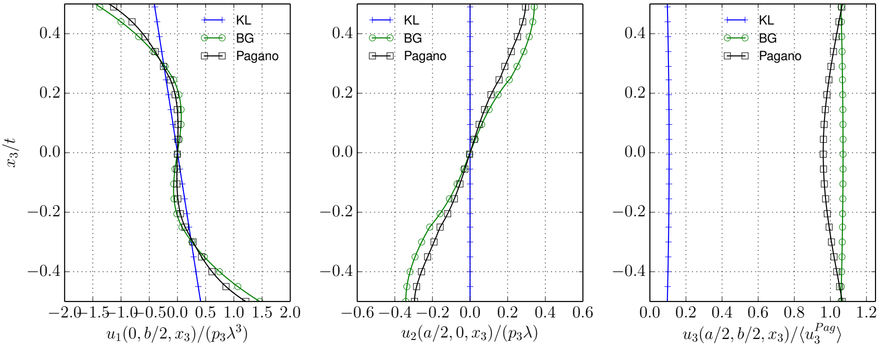

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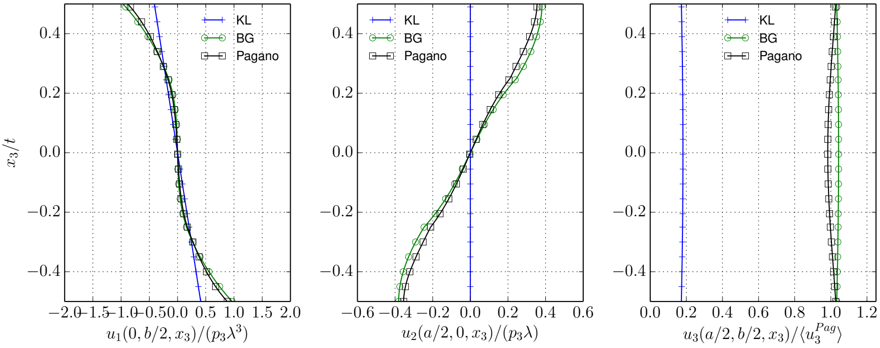

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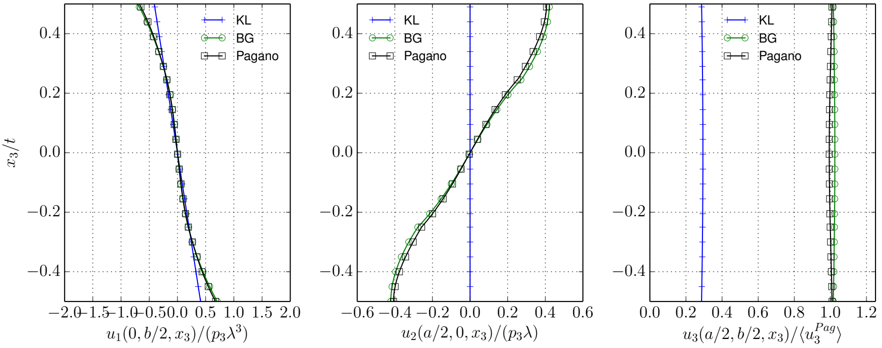

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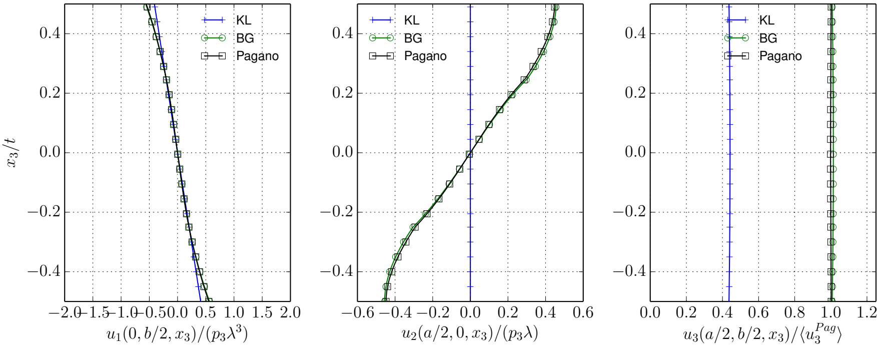

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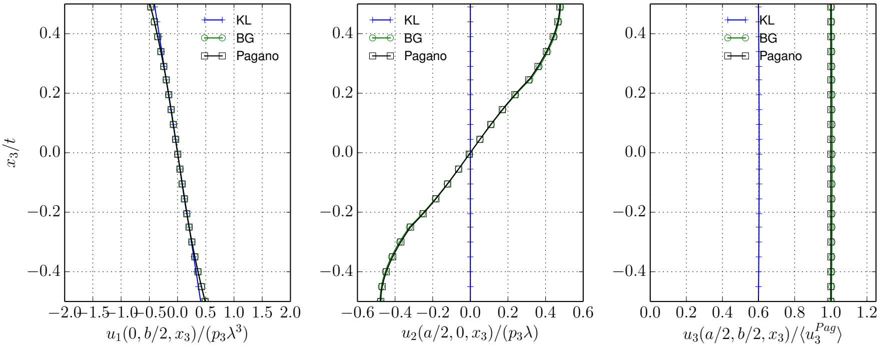

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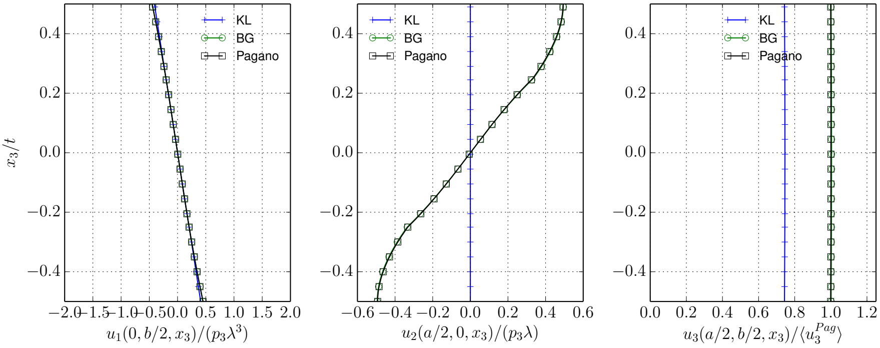

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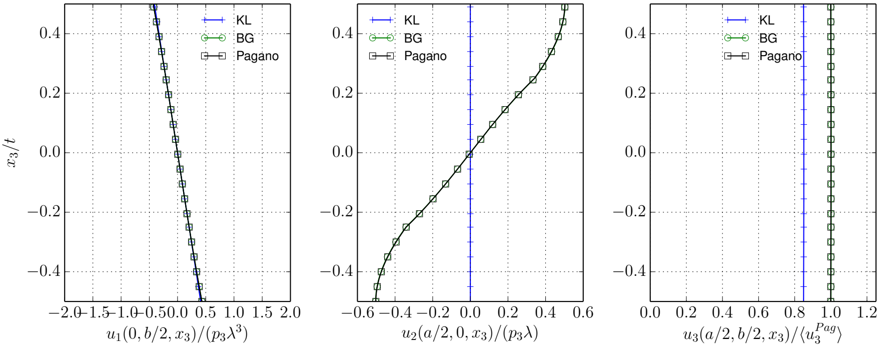

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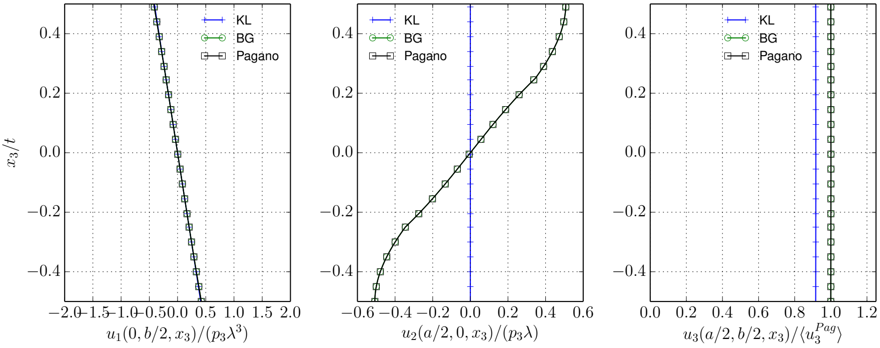
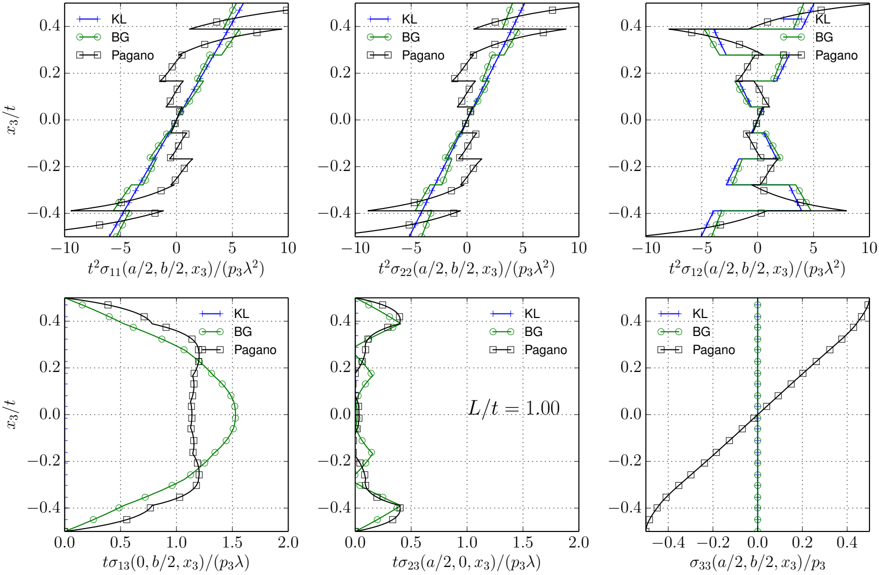
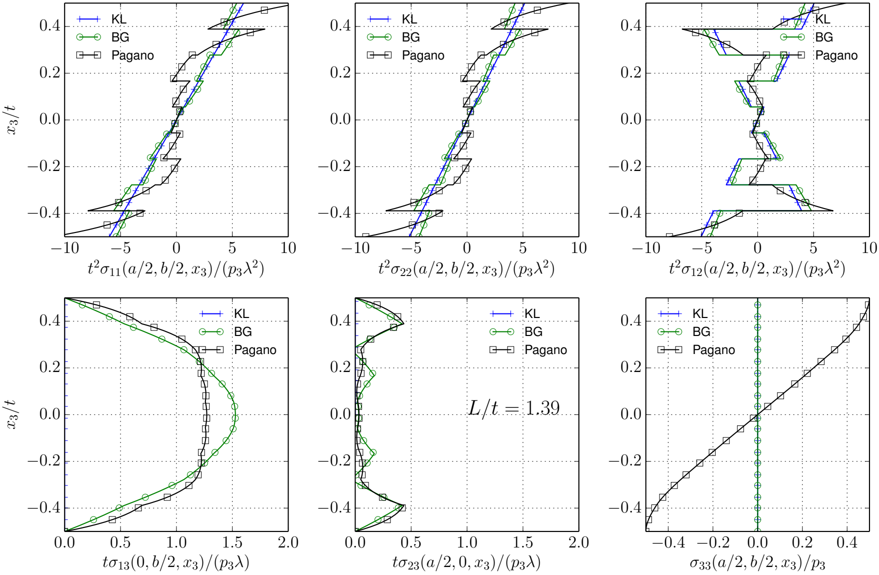


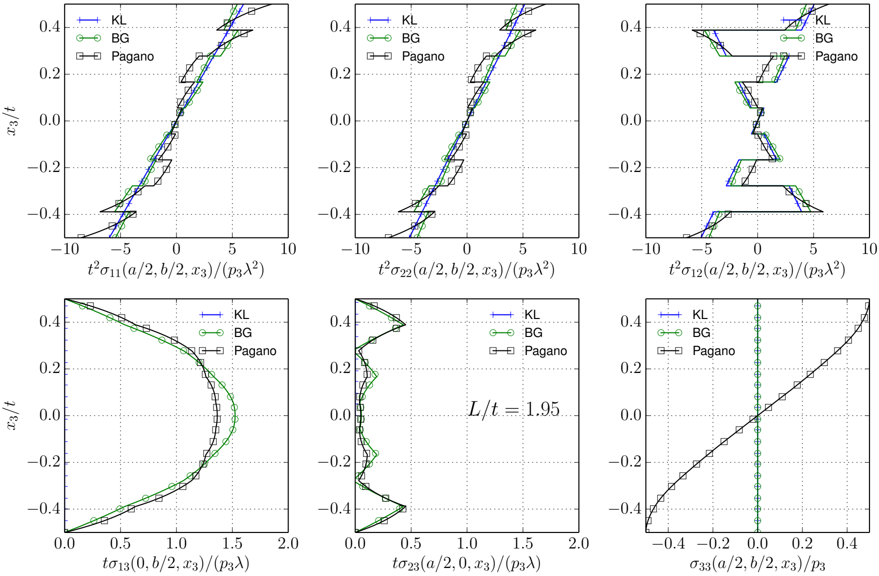
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



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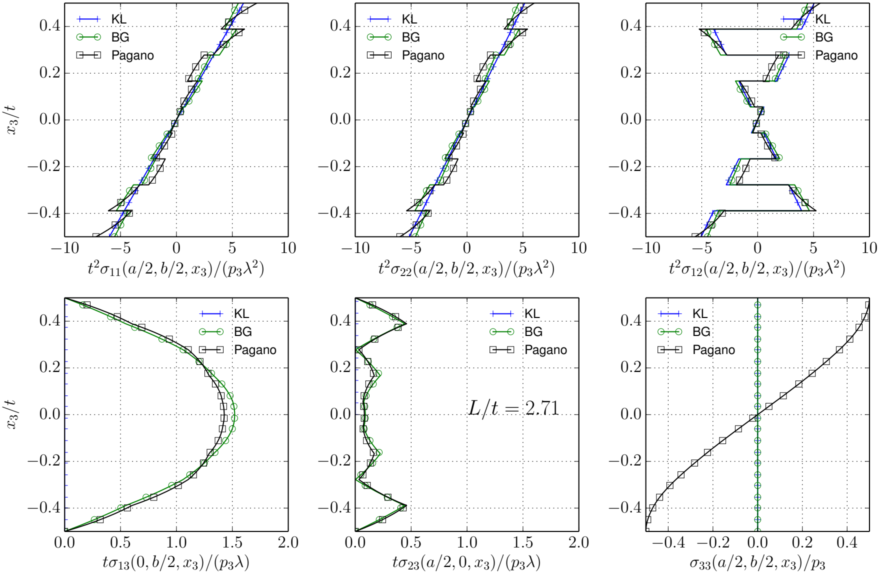

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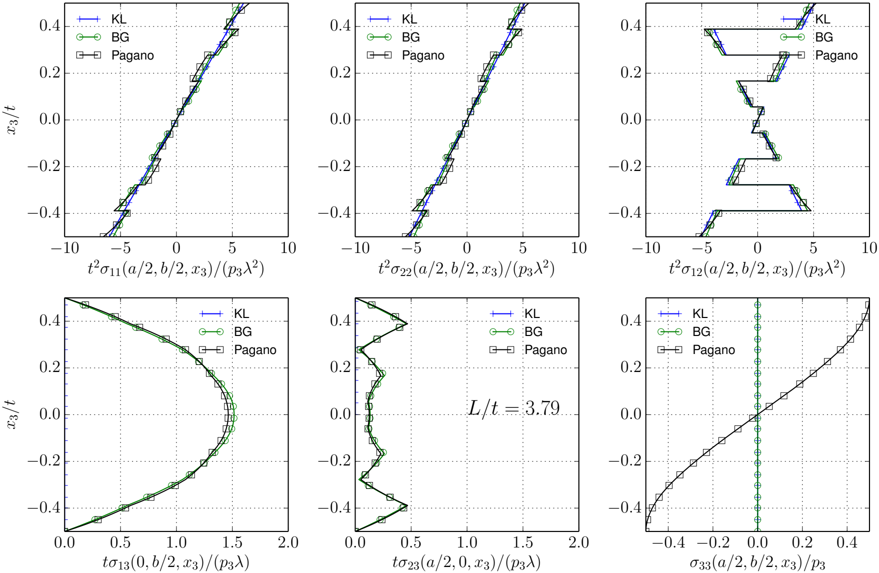

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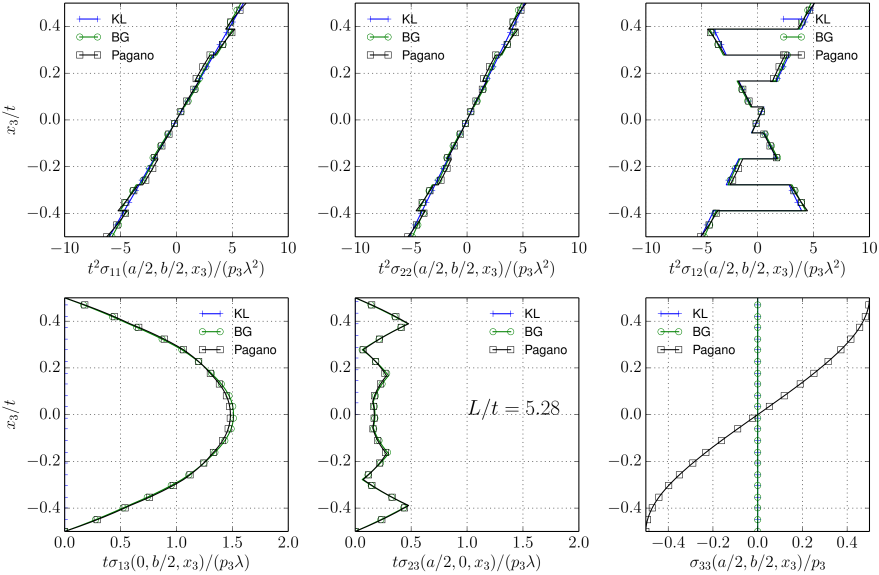

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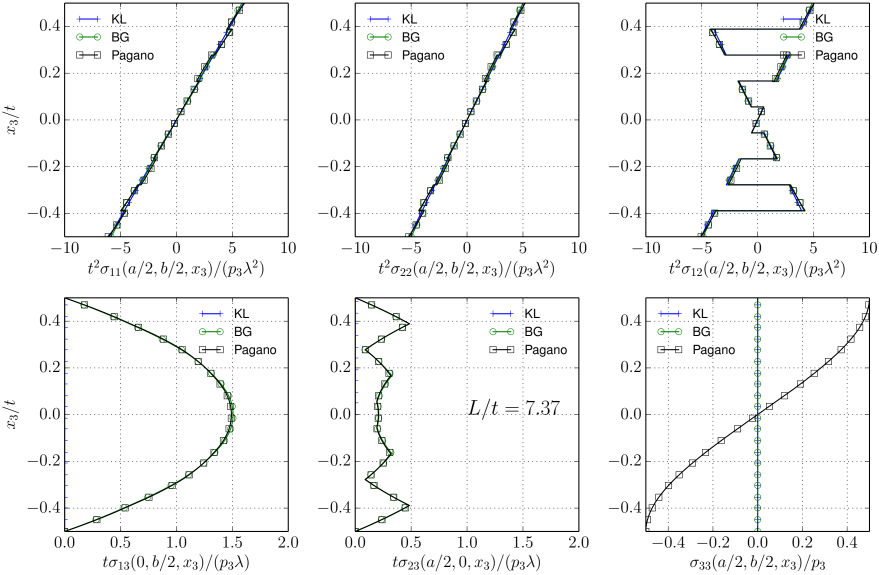

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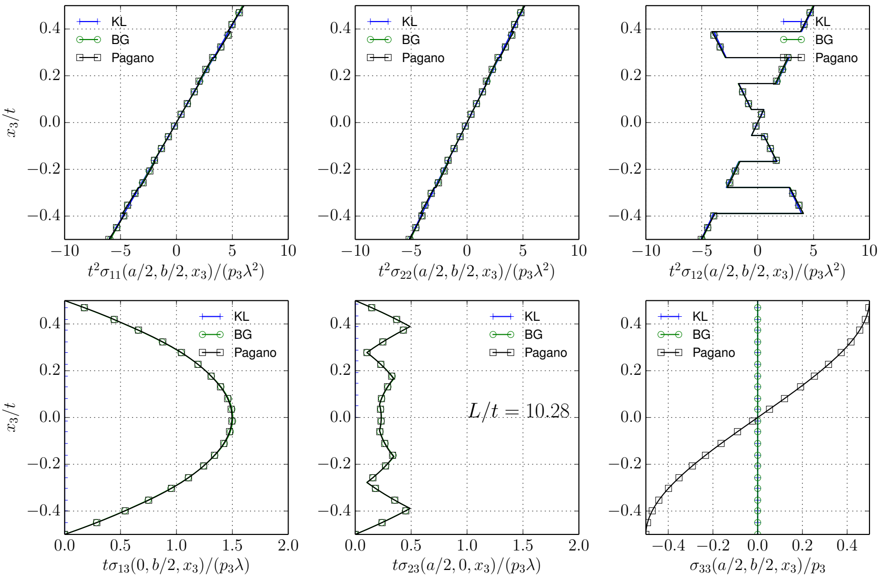

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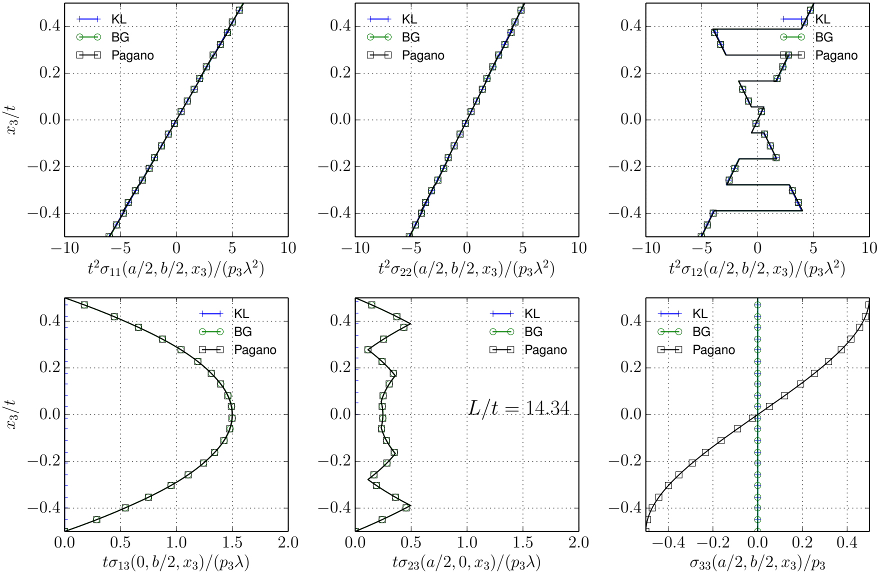
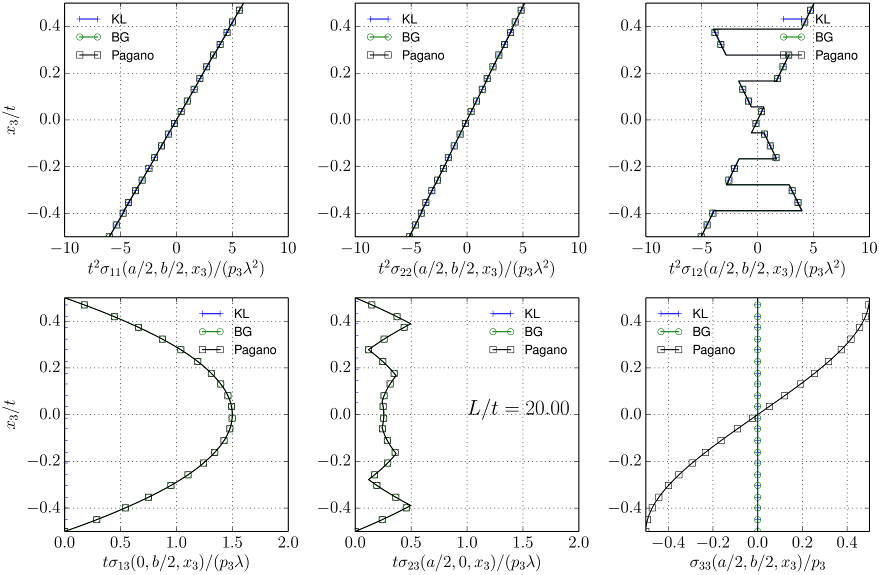
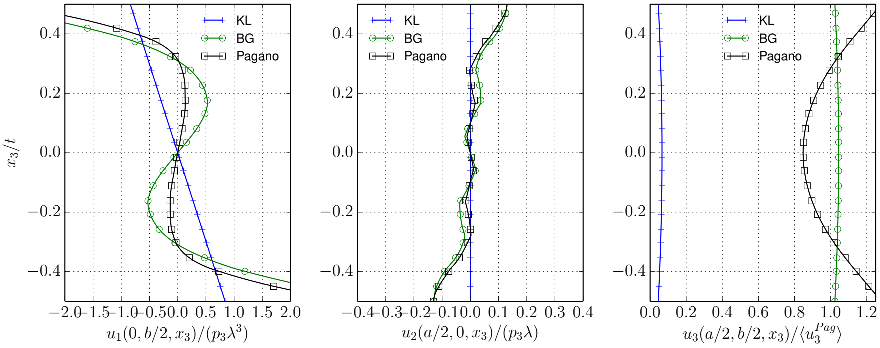


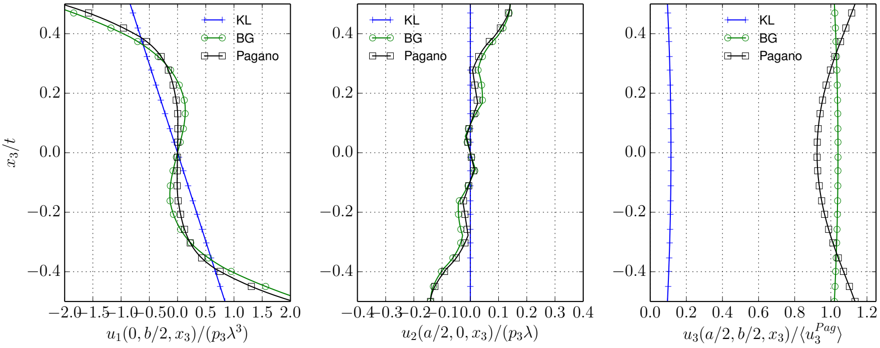
Stress distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



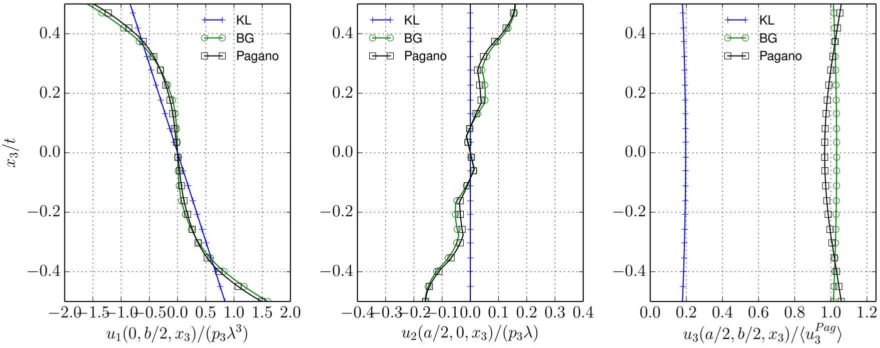
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



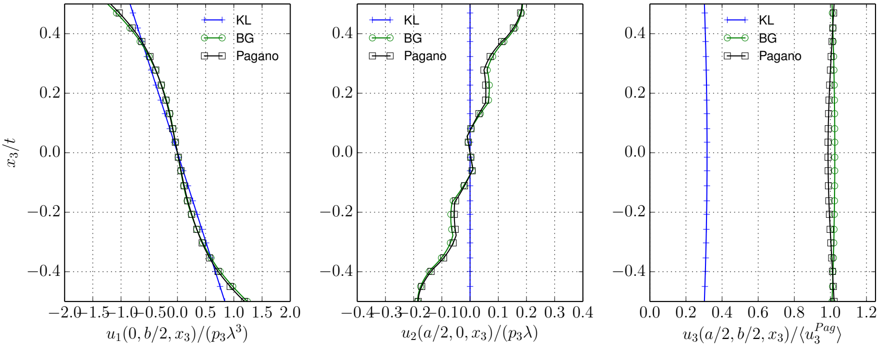
Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Displacement distributions for a $[45°, -45°]^4, 45°$ stack



Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



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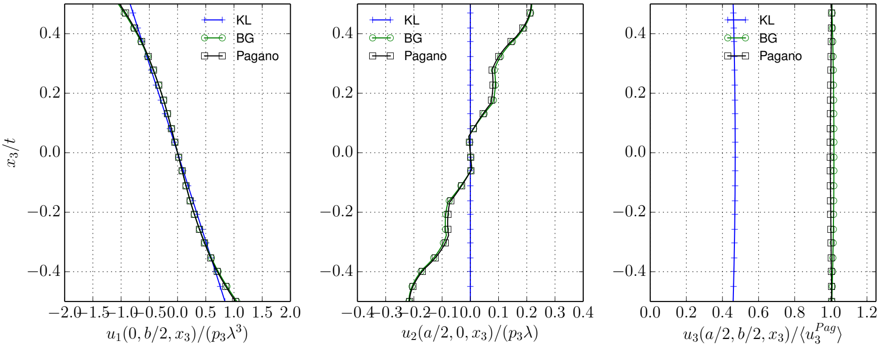
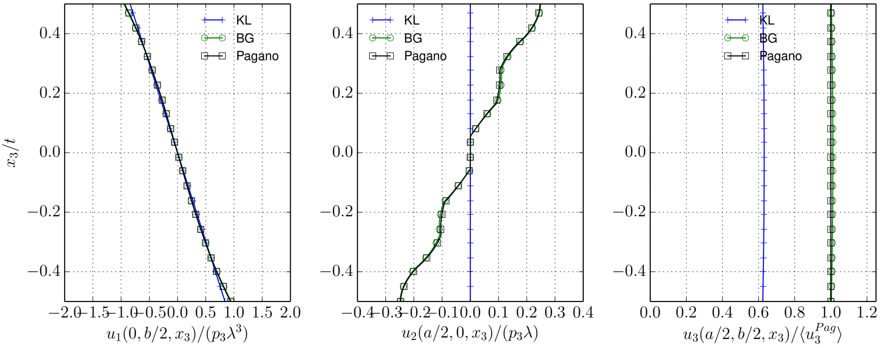


Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



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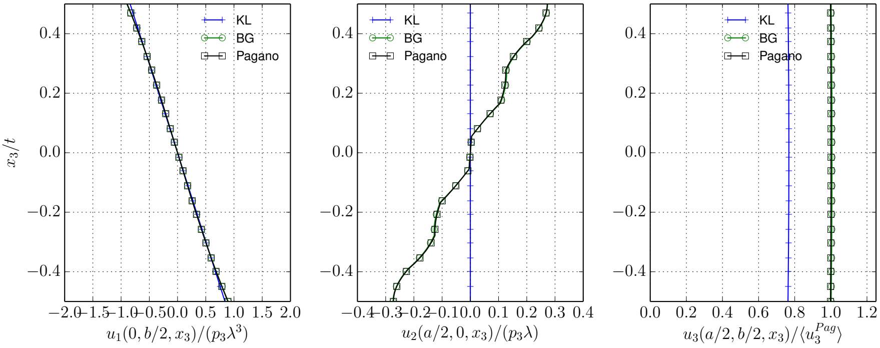

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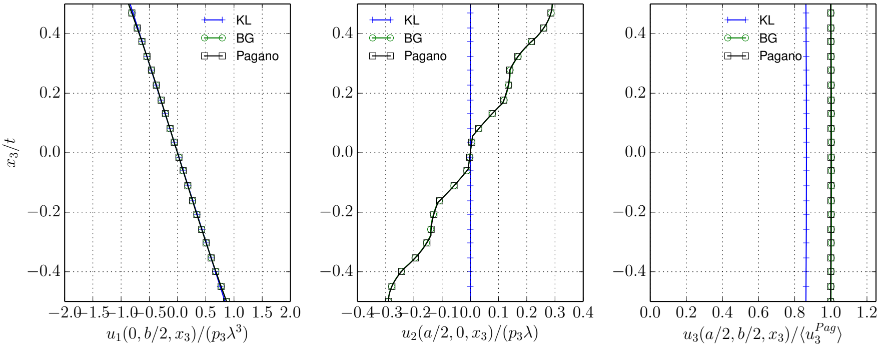

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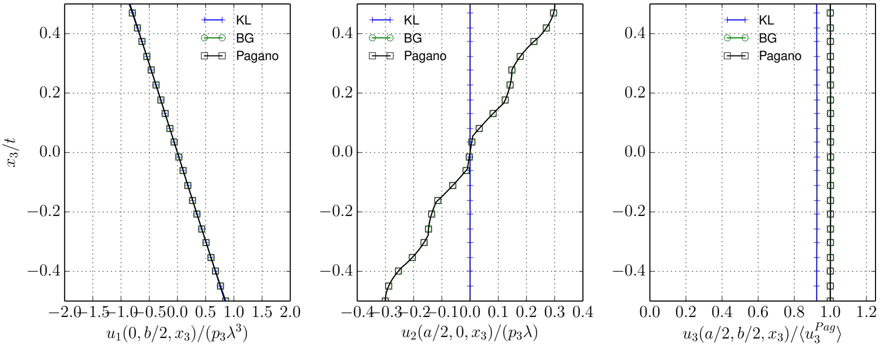
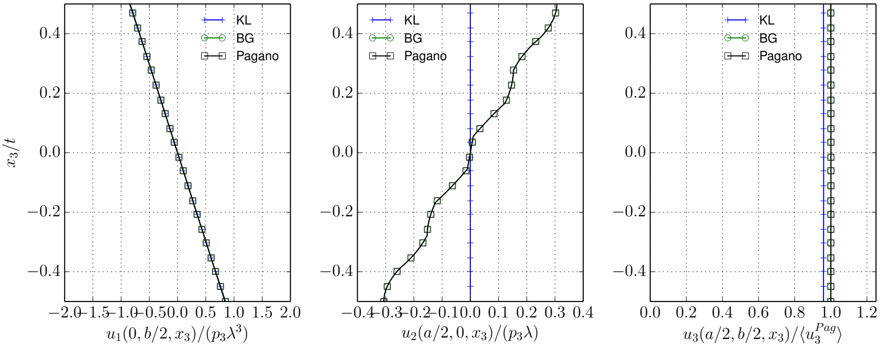
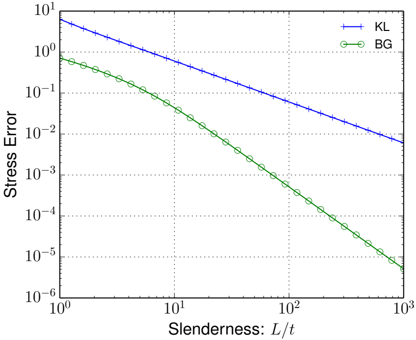
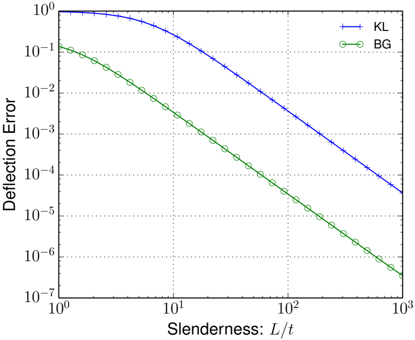


Displacement distributions for a $[45^\circ, -45^\circ]^4, 45^\circ$ stack



Convergence for a $[30°, -30°, 30°]$ stack





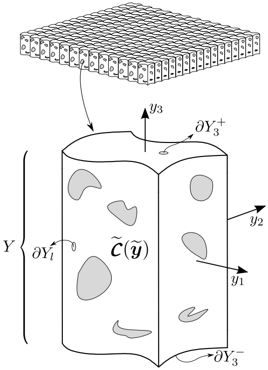
$\Delta(\sigma)$ rate: KL $\sim t$ and BG $\sim t^2$

$\Delta(U_3)$ rate: KL $\sim t^2$ and BG $\sim t^2$

Contents

Extension to periodic plates

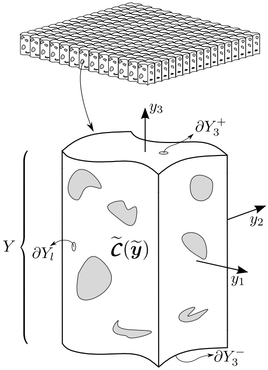
- Unit-cell and average estimates



Extension to periodic plates

- Unit-cell and average estimates
- Bending auxiliary problem (Caillerie, 1984)

$$\mathcal{P}^K \begin{cases} \underset{\sim}{\boldsymbol{\sigma}}^K \cdot \underline{\boldsymbol{\nabla}} = 0 \\ \underset{\sim}{\boldsymbol{\sigma}}^K = \underset{\approx}{\boldsymbol{C}}\left(\underline{\boldsymbol{y}}\right) : \underset{\sim}{\boldsymbol{\varepsilon}}^K \\ \underset{\sim}{\boldsymbol{\varepsilon}}^K = y_3 \underset{\sim}{\boldsymbol{K}} + \underline{\boldsymbol{\nabla}} \otimes^s \underline{\boldsymbol{u}}^{per} \\ \underset{\sim}{\boldsymbol{\sigma}}^K \cdot \underline{\boldsymbol{e}}_3 = 0 \text{ on free faces } \partial Y_3^{\pm} \\ \underset{\sim}{\boldsymbol{\sigma}}^K \cdot \underline{\boldsymbol{n}} \text{ skew-periodic on lateral edge } \partial Y_l \\ \underline{\boldsymbol{u}}^{per}(\underline{\boldsymbol{y}}) \ (y_1, y_2)\text{-periodic on lateral edge } \partial Y_l \end{cases}$$



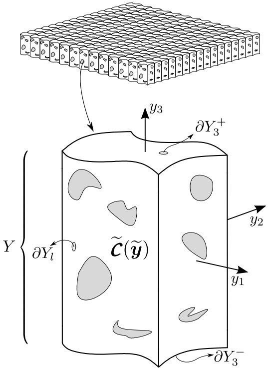
$\rightarrow$ gives:

Localization $\underline{\boldsymbol{u}}^K$ $\underset{\sim}{\boldsymbol{\sigma}}^K$ related to the curvature $\underset{\sim}{\boldsymbol{K}}$
Bending stiffness: $\underset{\approx}{\boldsymbol{D}}$

Extension to periodic plates

- Unit-cell and average estimates
- Bending auxiliary problem (Caillerie, 1984)
- Shear auxiliary problem

$$\mathcal{P}^R \begin{cases} \underset{\sim}{\boldsymbol{\sigma}}^R \cdot \underline{\boldsymbol{\nabla}} + \underset{\sim}{\boldsymbol{\sigma}}^M(\underline{\boldsymbol{y}}) = 0 \\ \underset{\sim}{\boldsymbol{\sigma}}^R = \underset{\approx}{\boldsymbol{C}}\left(\underline{\boldsymbol{y}}\right) : \left(\underset{\sim}{\boldsymbol{\delta}} \otimes^s \underline{\boldsymbol{u}}^M + \underline{\boldsymbol{\nabla}} \otimes^s \underline{\boldsymbol{u}}^R\right) \\ \underset{\sim}{\boldsymbol{\sigma}}^R \cdot \underline{\boldsymbol{e}}_3 = 0 \text{ on free faces } \partial Y_3^{\pm} \\ \underset{\sim}{\boldsymbol{\sigma}}^R \cdot \underline{\boldsymbol{n}} \text{ skew-periodic on lateral edge } \partial Y_l \\ \underline{\boldsymbol{u}}^R(\underline{\boldsymbol{y}}) \ (y_1, y_2)\text{-periodic on lateral edge } \partial Y_l \end{cases}$$



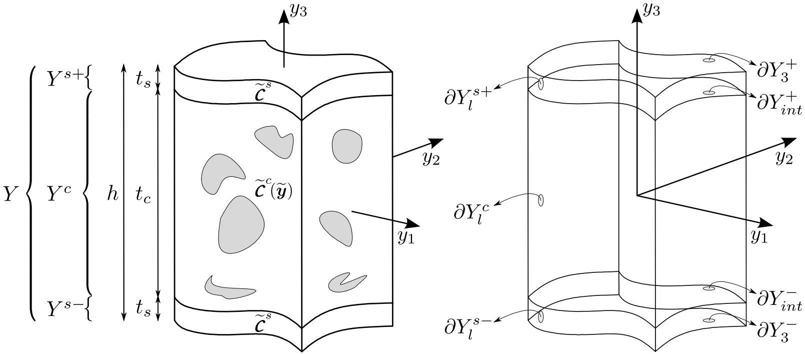
$\rightarrow$ gives:

Localization $\underline{\boldsymbol{u}}^R$ and $\underset{\sim}{\boldsymbol{\sigma}}^R$ related to $\underset{\frown}{\boldsymbol{R}}$
Shear compliance tensor: $\boldsymbol{f}$

Contents

Justification of the Sandwich Theory

- Divide in 3 layers
  (homogeneous skins and heterogeneous core)

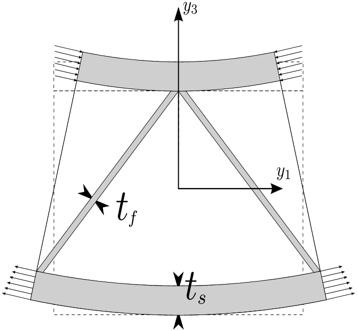


Justification of the Sandwich Theory

- Divide in 3 layers
  (homogeneous skins and heterogeneous core)
- Bending auxiliary problem
  - Contrast assumption $\Leftrightarrow t_f \ll t_s$:
    $\rightarrow t_s/t_f$ Contrast ratio

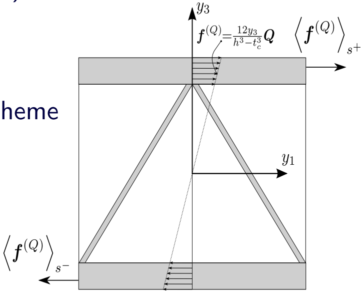
$\Rightarrow$ Skins under traction/compression

$\Rightarrow$ Core not involved in Bending stiffness



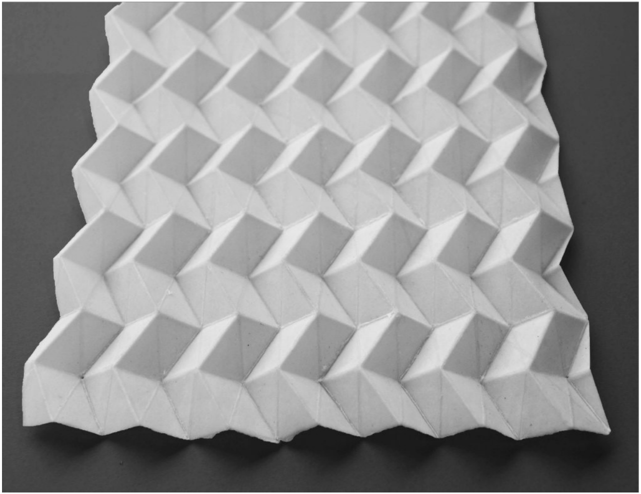
Justification of the Sandwich Theory

- Divide in 3 layers
  (homogeneous skins and heterogeneous core)
- Bending auxiliary problem
- Shear auxiliary problem
  - $\underline{\boldsymbol{f}}^R$ becomes $\underline{\boldsymbol{f}}^{(Q)}$ + Direct homogenization scheme
  - The BG is degenerated into RM model
  - $\underline{\boldsymbol{f}}^{(Q)}$ confirms the classical intuition
  - Proof of the bounds from Kelsey et al. (1958)



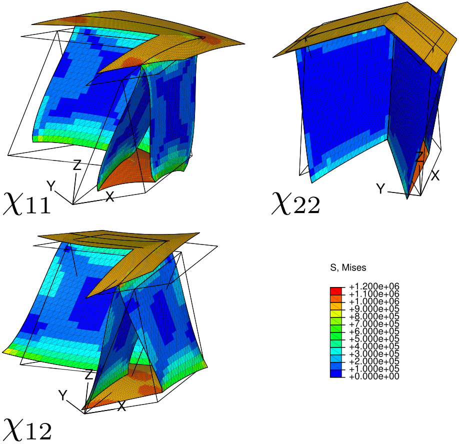
Lebée and Sab (2012a)

Application to the chevron pattern



Application to the chevron pattern

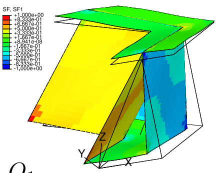
Bending:



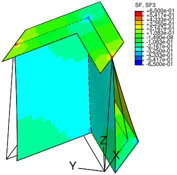
Application to the chevron pattern

Shear forces localization $\underset{\sim}{\boldsymbol{\sigma}}^{(Q)}$

- Overall shearing of the core



Q_1

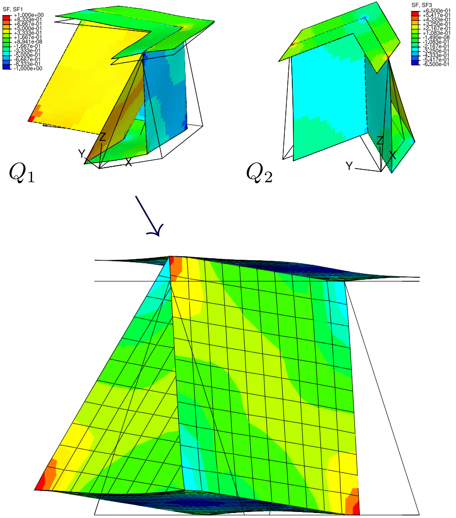


Q_2

Application to the chevron pattern

Shear forces localization $\underset{\sim}{\boldsymbol{\sigma}}^{(Q)}$

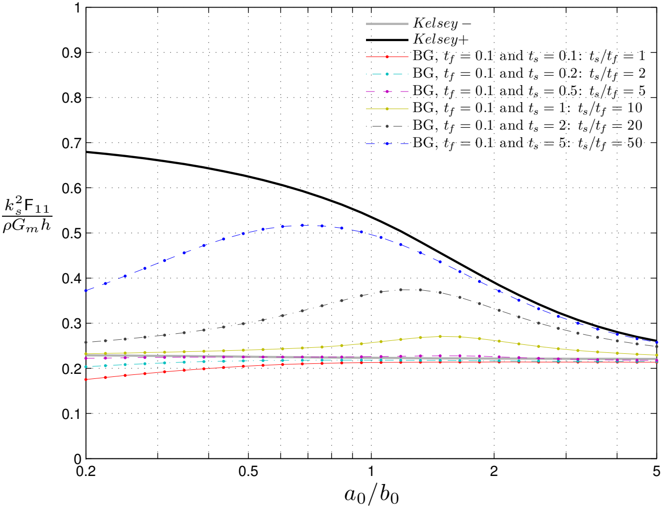
- Overall shearing of the core
- Out-of-plane skins distorsion



Application to the chevron pattern

Shear forces localization $\underset{\sim}{\boldsymbol{\sigma}}^{(Q)}$

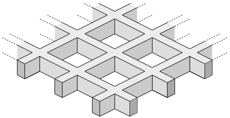
- Overall shearing of the core
- Out-of-plane skins distorsion
- Critically influence shear force stiffness



Lebée and Sab (2012b)

Contents

Homogenizing an orthogonal beam lattice?



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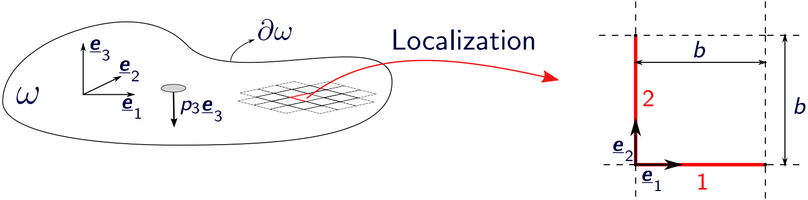
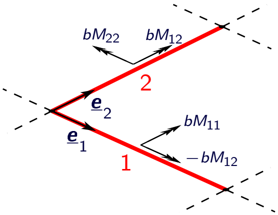
Thick-plate model (macro) + 2 St-Venant Beams (micro)



Field localization



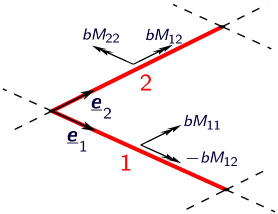
Bending moment $\left(\underline{r}^{(M)}, \underline{m}^{(M)}\right)$:

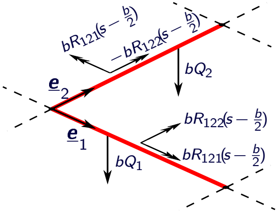
Apply $\underset{\sim}{M}$ "on average" on the unit-cell (Caillerie, 1984)

$$^{1}\underline{r}^{(M)} = {}^{2}\underline{r}^{(M)} = \underline{\mathbf{0}}$$

$$^{1}\underline{m}^{(M)} = \begin{pmatrix} -bM_{12} \\ bM_{11} \\ 0 \end{pmatrix}_1 \quad \text{and} \quad {}^{2}\underline{m}^{(M)} = \begin{pmatrix} bM_{12} \\ bM_{22} \\ 0 \end{pmatrix}_2$$

Field localization





Bending moment $\left(\underline{r}^{(M)}, \underline{m}^{(M)}\right)$:

Apply $\underset{\sim}{M}$ "on average" on the unit-cell (Caillerie, 1984)

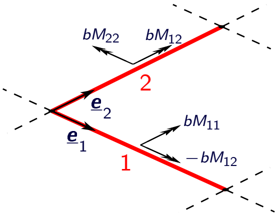
Bending gradient $\left(\underline{r}^{(R)}, \underline{m}^{(R)}\right)$:

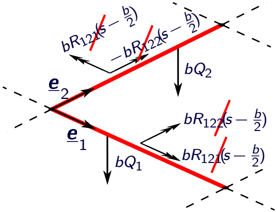
Assume $M_{\alpha\beta} = R_{\alpha\beta\gamma} x_\gamma$ (Lebée and Sab, 2013a)

$$^1\underline{r}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b\underbrace{(R_{111}+R_{122})}_{Q_1} \end{pmatrix}_1 \qquad ^1\underline{m}^{(R)} = \begin{pmatrix} bR_{121}\left(s-\frac{b}{2}\right) \\ bR_{122}\left(s-\frac{b}{2}\right) \\ 0 \end{pmatrix}_1$$

$$^2\underline{r}^{(R)} = \begin{pmatrix} 0 \\ 0 \\ b\underbrace{(R_{121}+R_{222})}_{Q_2} \end{pmatrix}_2 \qquad ^2\underline{m}^{(R)} = \begin{pmatrix} -bR_{122}\left(s-\frac{b}{2}\right) \\ bR_{121}\left(s-\frac{b}{2}\right) \\ 0 \end{pmatrix}_2$$

Field localization





Bending moment $\left(\underline{\boldsymbol{r}}^{(M)}, \underline{\boldsymbol{m}}^{(M)}\right)$:

Apply $\underset{\sim}{\boldsymbol{M}}$ "on average" on the unit-cell (Caillerie, 1984)

Bending gradient $\left(\underline{\boldsymbol{r}}^{(R)}, \underline{\boldsymbol{m}}^{(R)}\right)$:

Assume $M_{\alpha\beta} = R_{\alpha\beta\gamma} x_\gamma$ (Lebée and Sab, 2013a)

Reissner-Mindlin $\left(\underline{\boldsymbol{r}}^{(Q)}, \underline{\boldsymbol{m}}^{(Q)}\right)$:

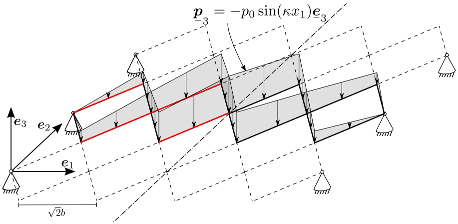
Assume cylindrical bending (Whitney, 1969; Cecchi and Sab, 2007)

$$Q_1 = R_{111},\ Q_2 = R_{222}, \quad R_{121} = R_{122} = R_{221} = R_{112} = 0$$

$${}^1\underline{\boldsymbol{r}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_1 \end{pmatrix}_1 \quad \text{and} \quad {}^1\underline{\boldsymbol{m}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_1$$

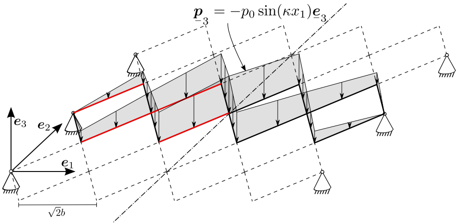
$${}^2\underline{\boldsymbol{r}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ bQ_2 \end{pmatrix}_2 \quad \text{and} \quad {}^2\underline{\boldsymbol{m}}^{(Q)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_2$$

Application: lattice rotated 45° and cylindrical bending

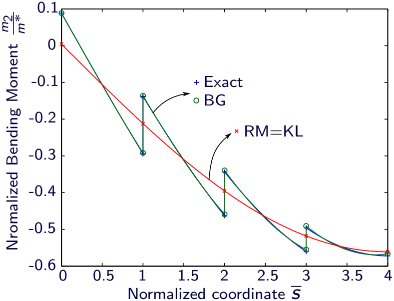


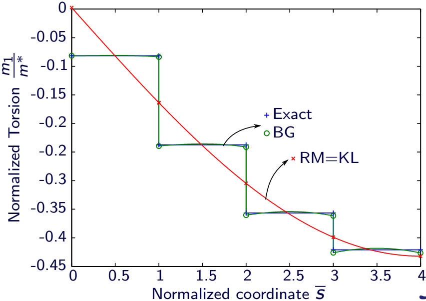
- Exact solution
- Plate solution + Localization (RM and BG)

Application: lattice rotated 45° and cylindrical bending



- Exact solution
- Plate solution + Localization (RM and BG)





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